

Primer for Differentiation

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After reading this chapter, you should be able to:

1. *understand the basics of differentiation,*
2. *relate the slopes of the secant line and tangent line to the derivative of a function,*
3. *find derivatives of polynomial, trigonometric and transcendental functions,*
4. *use rules of differentiation to differentiate functions,*
5. *find maxima and minima of a function, and*
6. *apply concepts of differentiation to real world problems.*

In this primer, we will review the concepts of differentiation you learned in calculus. Mostly those concepts are reviewed that are applicable in learning about numerical methods. These include the concepts of the secant line to learn about numerical differentiation of functions, the slope of a tangent line as a background to solving nonlinear equations using the Newton-Raphson method, finding maxima and minima of functions as a means of optimization, the use of the Taylor series to approximate functions, etc.

Introduction

The derivative of a function represents the rate of change of a variable with respect to another variable. For example, the velocity of a body is defined as the rate of change of the location of the body with respect to time. The location is the *dependent* variable while time is the *independent* variable. Now if we measure the rate of change of velocity with respect to time, we get the acceleration of the body. In this case, the velocity is the *dependent* variable while time is the *independent* variable.

Whenever differentiation is introduced to a student, two concepts of the secant line and tangent line (Figure 1) are revisited.

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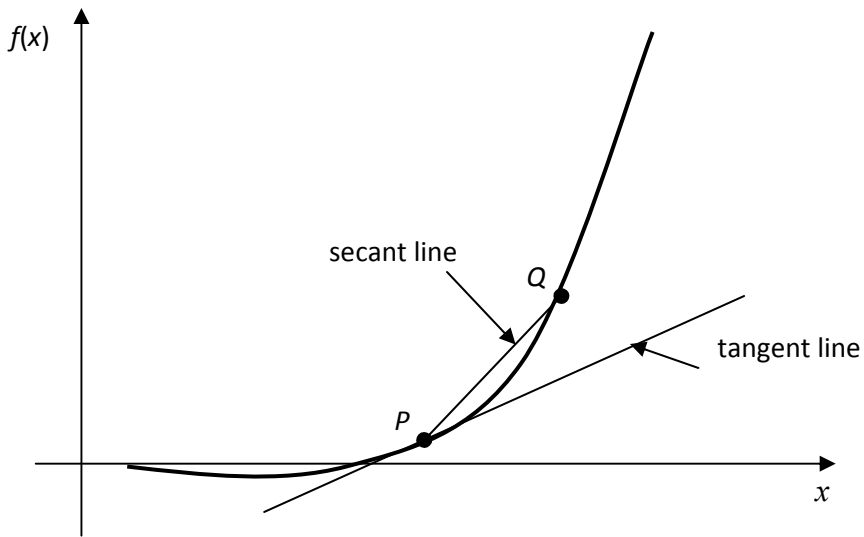


Figure 1 Function curve with tangent and secant lines.

Let P and Q be two points on the curve as shown in Figure 1. The secant line is the straight line drawn through P and Q .

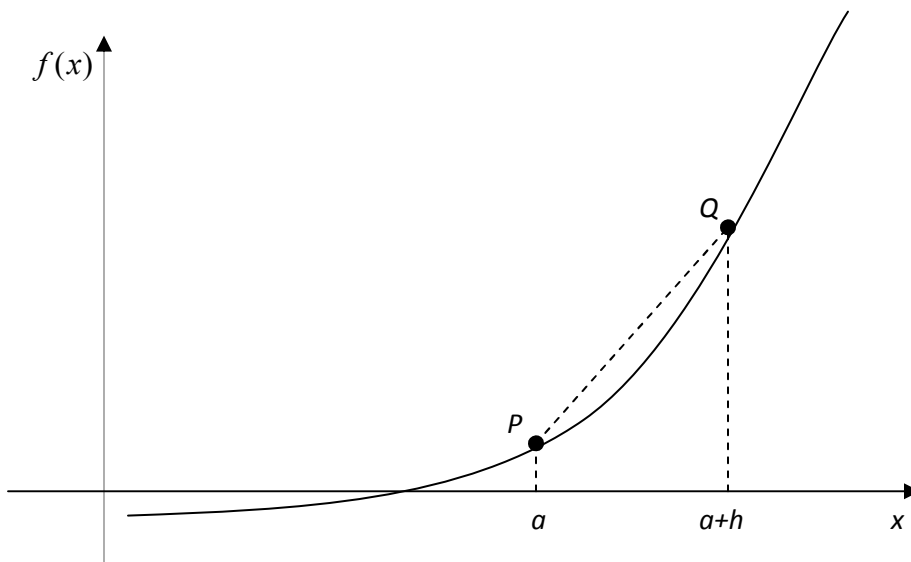


Figure 2 Calculation of the secant line.

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The slope of the secant line (Figure 2) is then given as

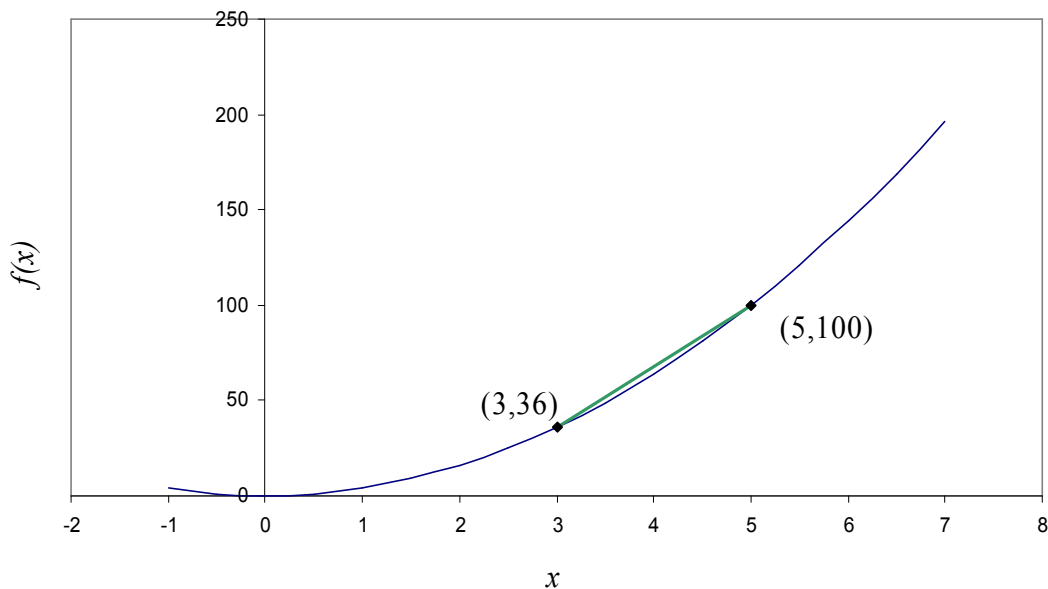
$$\begin{aligned} m_{PQ,\text{secant}} &= \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \frac{f(a+h) - f(a)}{h} \end{aligned}$$

As Q moves closer and closer to P , the limiting portion is called the tangent line. The slope of the tangent line $m_{PQ,\text{tangent}}$ then is the limiting value of $m_{PQ,\text{secant}}$ as $h \rightarrow 0$.

$$m_{PQ,\text{tangent}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Example 1

Find the slope of the secant line of the curve $y = 4x^2$ between points $(3,36)$ and $(5,100)$.



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Figure 3 Calculation of the secant line for the function $y = 4x^2$.

Solution

The slope of the secant line between (3,36) and (5,100) is

$$\begin{aligned} m &= \frac{f(5) - f(3)}{5 - 3} \\ &= \frac{100 - 36}{5 - 3} \\ &= 32 \end{aligned}$$

Example 2

Find the slope of the tangent line of the curve $y = 4x^2$ at point (3,36).

Solution

The slope of the tangent line at (3,36) is

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(3+h)^2 - 4(3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(9 + h^2 + 6h) - 36}{h} \\ &= \lim_{h \rightarrow 0} \frac{36 + 4h^2 + 24h - 36}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(4h + 24)}{h} \\ &= \lim_{h \rightarrow 0} (4h + 24) \\ &= 24 \end{aligned}$$



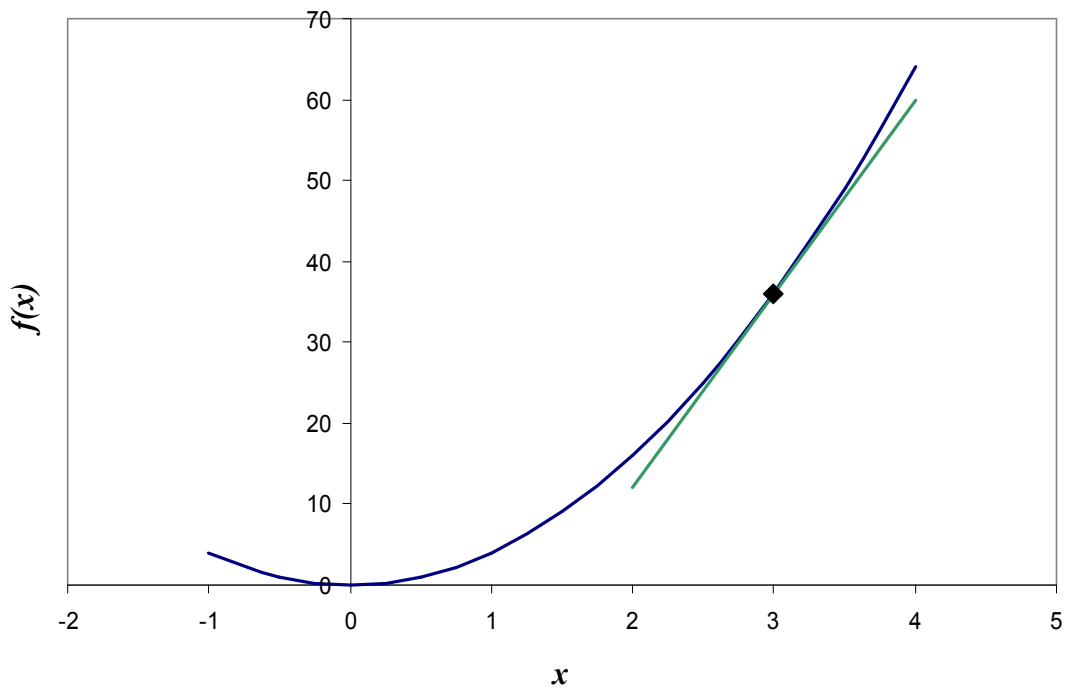


Figure 4 Calculation of the tangent line in the function $y = 4x^2$.

The slope of the tangent line is

$$\begin{aligned}
 m &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4(3+h)^2 - 4(3)^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{36 + 24h + 4h^2 - 36}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(24 + 4h)}{h} \\
 &= \lim_{h \rightarrow 0} (24 + 4h)
 \end{aligned}$$

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$$= 24$$

Derivative of a Function

Recall from calculus, the derivative of a function $f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Example 3

Find $f'(3)$ if $f(x) = 4x^2$.

Solution

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(3+h)^2 - 4(3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(9 + h^2 + 6h) - 36}{h} \\ &= \lim_{h \rightarrow 0} \frac{36 + 4h^2 + 24h - 36}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(4h + 24)}{h} \\ &= \lim_{h \rightarrow 0} (4h + 24) \\ &= 24 \end{aligned}$$

Example 4

Find $f'\left(\frac{\pi}{4}\right)$ if $f(x) = \sin(2x)$

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Solution

$$\begin{aligned}f'\left(\frac{\pi}{4}\right) &= \lim_{h \rightarrow 0} \frac{f\left(\frac{\pi}{4} + h\right) - f\left(\frac{\pi}{4}\right)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin\left(2\left(\frac{\pi}{4} + h\right)\right) - \sin\left(2\left(\frac{\pi}{4}\right)\right)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{2} + 2h\right) - \sin\left(\frac{\pi}{2}\right)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{2}\right)\cos(2h) + \cos\left(\frac{\pi}{2}\right)\sin(2h) - \sin\left(\frac{\pi}{2}\right)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(2h) + 0 - 1}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(2h) - 1}{h} \\&= 0\end{aligned}$$

from knowing that

$$\lim_{h \rightarrow 0} \frac{1 - \cos(h)}{h} = 0$$

Second Definition of Derivatives

There is another form of the definition of the derivative of a function. The derivative of the function $f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

As $x \rightarrow a$, the definition is nothing but the slope of the tangent line at P .

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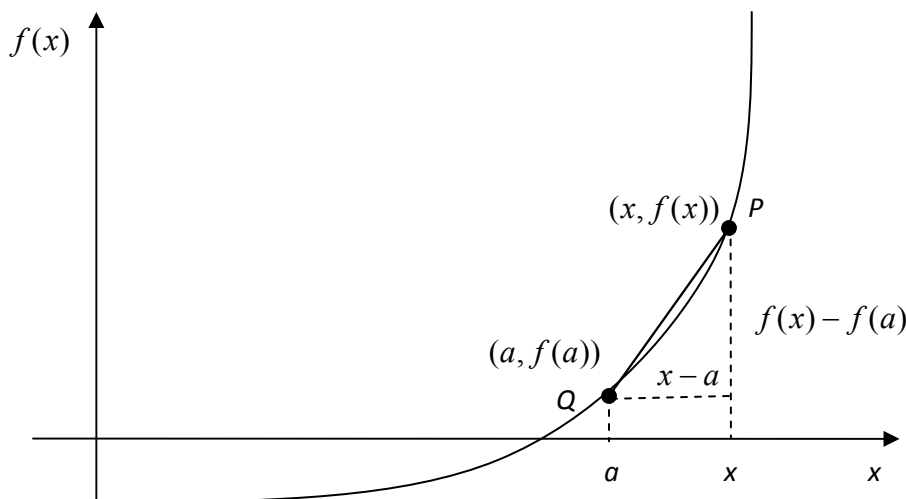


Figure 5 Graph showing the second definition of the derivative.

Example 5

Find $f'(3)$ if $f(x) = 4x^2$ by using the form

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

of the definition of a derivative.

Solution

$$\begin{aligned} f'(3) &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{4x^2 - 4(3)^2}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{4x^2 - 36}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{4(x^2 - 9)}{x - 3} \end{aligned}$$

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$$\begin{aligned}
&= \lim_{x \rightarrow 3} \frac{4(x-3)(x+3)}{x-3} \\
&= \lim_{x \rightarrow 3} 4(x+3) \\
&= 4(3+3) \\
&= 24
\end{aligned}$$

Finding equations of a tangent line

One of the numerical methods used to solve a nonlinear equation is called the *Newton-Raphson method*. This method is based on the knowledge of finding the tangent line to a curve at a point. Let us look at an example to illustrate finding the equation of the tangent line to a curve.

Example 6

Find the equation of the line tangent to the function

$$f(x) = x^3 - 0.165x + 3.993 \times 10^{-4} \text{ at } x = 0.05.$$

Solution

The line tangent is a straight line of the form

$$y = mx + c$$

To find the equation of the tangent line, let us first find the slope m of the straight line.

$$f'(x) = 3x^2 - 0.165$$

$$f'(0.05) = 3(0.05)^2 - 0.165$$

$$= -0.1575$$

$$m = -0.1575$$

To find the value of the y -intercept c of the straight line, we first find the value of the function at $x = 0.05$.



$$\begin{aligned}
 f(0.05) &= (0.05)^3 - 0.165(0.05) + 3.993 \times 10^{-4} \\
 &= -0.0077257
 \end{aligned}$$

The tangent line passes through the point $(0.05, -0.0077257)$, so

$$\begin{aligned}
 -0.0077257 &= m(0.05) + c \\
 -0.0077257 &= -0.1575(0.05) + c \\
 c &= 0.0001493
 \end{aligned}$$

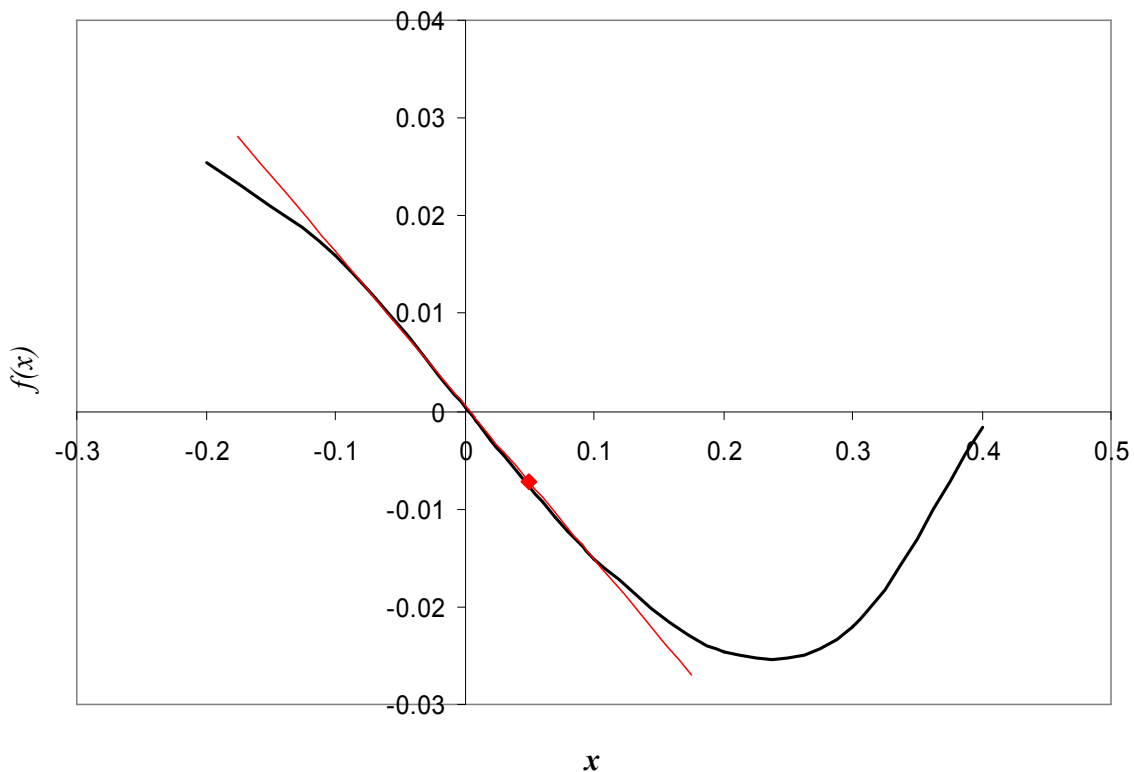


Figure 6 Graph of function $f(x)$ and the tangent line at $x = 0.05$.

Hence,

$$y = mx + c$$

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$$= -0.1575x + 0.0001493$$

is the equation of the tangent line.

Other Notations of Derivatives

Derivates can be denoted in several ways. For the first derivative, the notations are

$$f'(x), \frac{d}{dx}f(x), y', \text{ and } \frac{dy}{dx}$$

For the second derivative, the notations are

$$f''(x), \frac{d^2}{dx^2}f(x), y'', \text{ and } \frac{d^2y}{dx^2}$$

For the n^{th} derivative, the notations are

$$f^{(n)}(x), \frac{d^n}{dx^n}f(x), y^{(n)}, \frac{d^n y}{dx^n}$$

Theorems of Differentiation

Several theorems of differentiation are given to show how one can find the derivative of different functions.

Theorem 1

The derivative of a constant is zero. If $f(x) = k$, where k is a constant, $f'(x) = 0$.

Example 7

Find the derivative of $f(x) = 6$.

Solution

$$f(x) = 6$$



$$f'(x) = 0$$

Theorem 2

The derivative of $f(x) = x^n$, where $n \neq 0$ is $f'(x) = nx^{n-1}$.

Example 8

Find the derivative of $f(x) = x^6$.

Solution

$$f(x) = x^6$$

$$f'(x) = 6x^{6-1}$$

$$= 6x^5$$

Example 9

Find the derivative of $f(x) = x^{-6}$.

Solution

$$f(x) = x^{-6}$$

$$f'(x) = -6x^{-6-1}$$

$$= -6x^{-7}$$

$$= -\frac{6}{x^7}$$

Theorem 3

The derivative of $f(x) = kg(x)$, where k is a constant is $f'(x) = kg'(x)$.

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Example 10

Find the derivative of $f(x) = 10x^6$.

Solution

$$f(x) = 10x^6$$

$$f'(x) = \frac{d}{dx}(10x^6)$$

$$= 10 \frac{d}{dx} x^6$$

$$= 10(6x^5)$$

$$= 60x^5$$

Theorem 4

The derivative of $f(x) = u(x) \pm v(x)$ is $f'(x) = u'(x) \pm v'(x)$.

Example 11

Find the derivative of $f(x) = 3x^3 + 8$.

Solution

$$f(x) = 3x^3 + 8$$

$$f'(x) = \frac{d}{dx}(3x^3 + 8)$$

$$= \frac{d}{dx}(3x^3) + \frac{d}{dx}(8)$$

$$= 3 \frac{d}{dx}(x^3) + 0$$

$$= 3(3x^2)$$

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$$= 9x^2$$

Theorem 5

The derivative of

$$f(x) = u(x)v(x)$$

is

$$f'(x) = u(x) \frac{d}{dx} v(x) + v(x) \frac{d}{dx} u(x). \quad (\text{Product Rule})$$

Example 12

Find the derivative of $f(x) = (2x^2 - 6)(3x^3 + 8)$

Solution

Using the product rule as given by Theorem 5 where,

$$f(x) = u(x)v(x)$$

$$f'(x) = u(x) \frac{d}{dx} v(x) + v(x) \frac{d}{dx} u(x)$$

$$f(x) = (2x^2 - 6)(3x^3 + 8)$$

$$u(x) = 2x^2 - 6$$

$$v(x) = 3x^3 + 8$$

Taking the derivative of $u(x)$,

$$\frac{du}{dx} = \frac{d}{dx} (2x^2 - 6)$$

$$= \frac{d}{dx} (2x^2) - \frac{d}{dx} (6)$$

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$$\begin{aligned}
&= 2 \frac{d}{dx}(x^2) - 0 \\
&= 2(2x) \\
&= 4x
\end{aligned}$$

Taking the derivative of $v(x)$,

$$\begin{aligned}
\frac{dv}{dx} &= \frac{d}{dx}(3x^3 + 8) \\
&= \frac{d}{dx}(3x^3) + \frac{d}{dx}(8) \\
&= 3 \frac{d}{dx}(x^3) + 0 \\
&= 3(3x^2) \\
&= 9x^2
\end{aligned}$$

Using the formula for the product rule

$$\begin{aligned}
f'(x) &= u(x) \frac{d}{dx} v(x) + v(x) \frac{d}{dx} u(x) \\
&= (2x^2 - 6)(9x^2) + (3x^3 + 8)(4x) \\
&= 18x^4 - 54x^2 + 12x^4 + 32x \\
&= 30x^4 - 54x^2 + 32x
\end{aligned}$$

Theorem 6

The derivative of

$$f(x) = \frac{u(x)}{v(x)}$$

is



$$f'(x) = \frac{v(x) \frac{d}{dx} u(x) - u(x) \frac{d}{dx} v(x)}{(v(x))^2} \quad (\text{Quotient Rule})$$

Example 13

Find the derivative of $f(x) = \frac{(2x^2 - 6)}{(3x^3 + 8)}$.

Solution

Use the quotient rule of Theorem 6, if

$$f(x) = \frac{u(x)}{v(x)}$$

then

$$f'(x) = \frac{v(x) \frac{d}{dx} u(x) - u(x) \frac{d}{dx} v(x)}{(v(x))^2}$$

From

$$f(x) = \frac{(2x^2 - 6)}{(3x^3 + 8)}$$

we have

$$u(x) = 2x^2 - 6$$

$$v(x) = 3x^3 + 8$$

Taking the derivative of $u(x)$,

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx} (2x^2 - 6) \\ &= \frac{d}{dx} (2x^2) - \frac{d}{dx} (6) \end{aligned}$$



$$\begin{aligned}
&= 2 \frac{d}{dx}(x^2) - 0 \\
&= 2(2x) \\
&= 4x
\end{aligned}$$

Taking the derivative of $v(x)$,

$$\begin{aligned}
\frac{dv}{dx} &= \frac{d}{dx}(3x^3 + 8) \\
&= \frac{d}{dx}(3x^3) + \frac{d}{dx}(8) \\
&= 3 \frac{d}{dx}(x^3) + 0 \\
&= 3(3x^2) \\
&= 9x^2
\end{aligned}$$

Using the formula for the quotient rule,

$$\begin{aligned}
f'(x) &= \frac{(3x^3 + 8)(4x) - (2x^2 - 6)(9x^2)}{(3x^3 + 8)^2} \\
&= \frac{12x^4 + 32x - 18x^4 + 54x^2}{9x^6 + 48x^3 + 64} \\
&= \frac{-6x^4 + 54x^2 + 32x}{9x^6 + 48x^3 + 64}
\end{aligned}$$

Table of Derivatives

$f(x)$	$f'(x)$
$x^n, n \neq 0$	nx^{n-1}
$kx^n, n \neq 0$	knx^{n-1}

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$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$1 - \tanh^2(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\csc(x)$	$-\csc(x)\cot(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
$\operatorname{csch}(x)$	$-\operatorname{coth}(x)\operatorname{csch}(x)$
$\operatorname{sech}(x)$	$-\tanh(x)\operatorname{sech}(x)$
$\operatorname{coth}(x)$	$1 - \operatorname{coth}^2(x)$
$\csc^{-1}(x)$	$-\frac{ x }{x^2\sqrt{x^2-1}}$
$\sec^{-1}(x)$	$\frac{ x }{x^2\sqrt{x^2-1}}$

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$\cot^{-1}(x)$	$\frac{-1}{1+x^2}$
a^x	$\ln(a)a^x$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$
e^x	e^x

Chain Rule of Differentiation

Sometimes functions that need to be differentiated do not fall in the form of simple functions or the forms described previously. Such functions can be differentiated using the chain rule if they are of the form $f(g(x))$. The chain rule states

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

For example, to find $f'(x)$ of $f(x) = (3x^2 - 2x)^4$, one could use the chain rule.

$$g(x) = (3x^2 - 2x)$$

$$g'(x) = 6x - 2$$

$$f'(g(x)) = 4(g(x))^3$$

$$\frac{d}{dx}((3x^2 - 2x)^4) = 4(3x^2 - 2x)^3(6x - 2)$$

Implicit Differentiation

Sometimes, the function to be differentiated is not given explicitly as an expression of the independent variable. In such cases, how do we find the derivatives? We will discuss this via examples.

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Example 14

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 2xy$

Solution

$$x^2 + y^2 = 2xy$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2xy)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2xy)$$

$$2x + 2y \frac{dy}{dx} = 2x \frac{dy}{dx} + 2y$$

$$2y \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - 2x$$

$$(2y - 2x) \frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx} = \frac{2y - 2x}{2y - 2x}$$

$$\frac{dy}{dx} = 1$$

Example 15

If $x^2 - xy + y^2 = 5$, find the value of y' .

Solution

$$x^2 - xy + y^2 = 5$$

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

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$$2x - x \frac{dy}{dx} - y + 2y \frac{dy}{dx} = 0$$

$$(-x + 2y) \frac{dy}{dx} = -2x + y$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

$$y' = \frac{y - 2x}{2y - x}$$

Higher order derivatives

So far, we have limited our discussion to calculating first derivative, $f'(x)$ of a function $f(x)$. What if we are asked to calculate higher order derivatives of $f(x)$.

A simple example of this is finding acceleration of a body from a function that gives the location of the body as a function of time. The derivative of the location with respect to time is the velocity of the body, followed by the derivative of velocity with respect to time being the acceleration. Hence, the second derivative of the location function gives the acceleration function of the body.

Example 16

Given $f(x) = 3x^3 - 2x - 7$, find the second derivative, $f''(x)$ and the third derivative, $f'''(x)$.

Solution

Given

$$f(x) = 3x^3 - 2x - 7$$

we have

$$\begin{aligned} f'(x) &= 3(3x^2) - 2 \\ &= 9x^2 - 2 \end{aligned}$$

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$$\begin{aligned}
 f''(x) &= \frac{d}{dx}(f'(x)) \\
 &= \frac{d}{dx}(9x^2 - 2) \\
 &= 9(2x) \\
 &= 18x
 \end{aligned}$$

$$\begin{aligned}
 f'''(x) &= \frac{d}{dx}(f''(x)) \\
 &= \frac{d}{dx}(18x) \\
 &= 18
 \end{aligned}$$

Example 17

If $x^2 - xy + y^2 = 5$, find the value of y'' .

Solution

From Example 15 we obtain

$$y' = \frac{y - 2x}{2y - x},$$

$$(2y - x)y' = y - 2x$$

$$\frac{d}{dx}((2y - x)y') = \frac{d}{dx}(y - 2x)$$

$$(2y - x) \frac{d}{dx}(y') + y' \frac{d}{dx}(2y - x) = \frac{d}{dx}(y) - \frac{d}{dx}(2x)$$

$$y''(2y - x) + y'(2y' - 1) = y' - 2$$

$$y'' = \frac{2y' - 2 - 2y'^2}{2y - x}$$

After substitution of y' ,



$$y'' = \frac{2\frac{y-2x}{2y-x} - 2 - 2\left(\frac{y-2x}{2y-x}\right)^2}{2y-x}$$

$$= -\frac{6(y^2 - xy + x^2)}{(2y-x)^3}$$

Finding maximum and minimum of a function

The knowledge of first derivative and second derivative of a function is used to find the minimum and maximum of a function. First, let us define what the maximum and minimum of a function are. Let $f(x)$ be a function in domain D , then

$f(a)$ is the maximum of the function if $f(a) \geq f(x)$ for all values of x in the domain D .

$f(a)$ is the minimum of the function if $f(a) \leq f(x)$ for all values of x in the domain D .

The minimum and maximum of a function are also the critical values of a function.

An extreme value can occur in the interval $[c, d]$ at

end points $x = c, x = d$.

a point in $[c, d]$ where $f'(x) = 0$.

a point in $[c, d]$ where $f'(x)$ does not exist.

These critical points can be the local maximas and minimas of the function (See Figure 8).

Example 18

Find the minimum and maximum value of $f(x) = x^2 - 2x - 5$ in the interval $[0, 5]$.



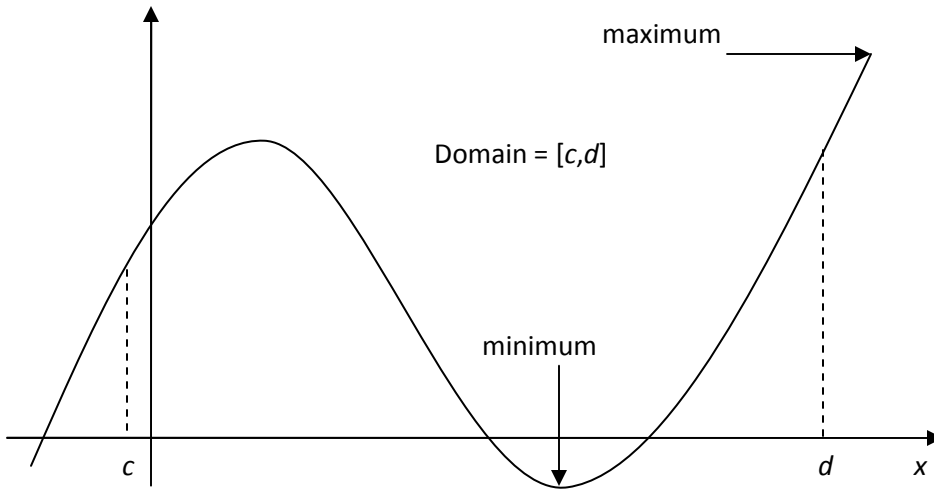


Figure 7 Graph illustrating the concepts of maximum and minimum.

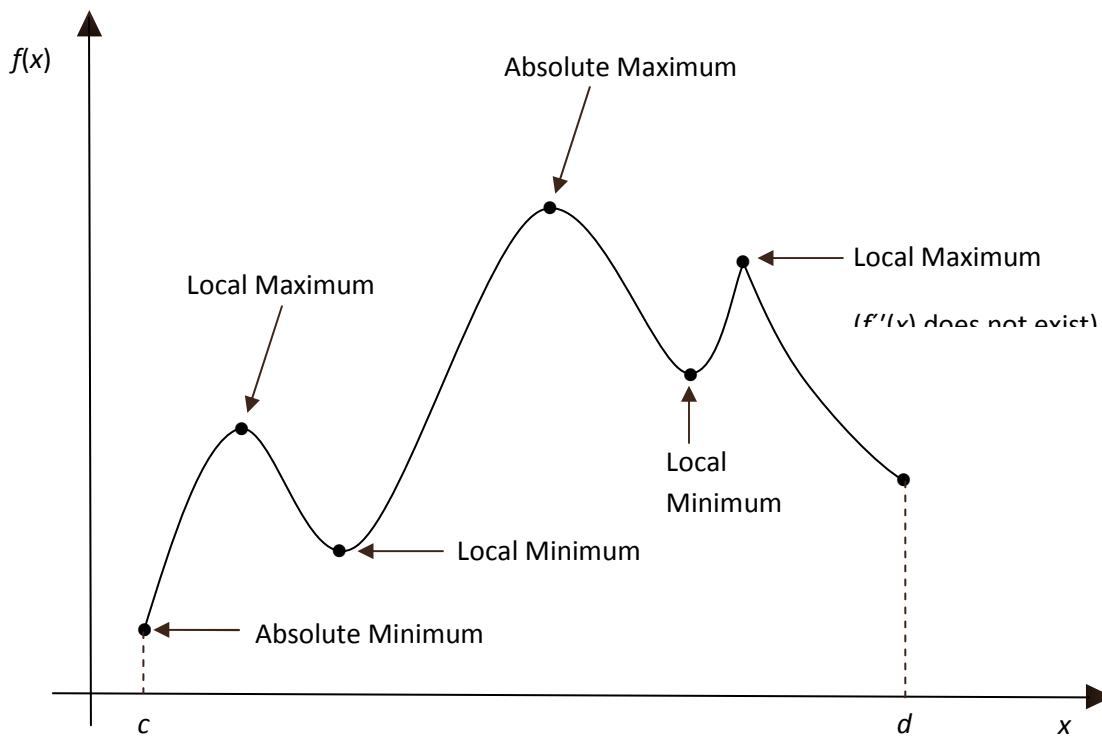


Figure 8 The plot shows critical points of $f(x)$ in $[c, d]$.

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Solution

$$f(x) = x^2 - 2x - 5$$

$$f'(x) = 2x - 2$$

$$f'(x) = 0 \text{ at } x = 1.$$

$f'(x)$ exists everywhere in $[0,5]$.

So the critical points are $x = 0$, $x = 1$, $x = 5$.

$$f(0) = (0)^2 - 2(0) - 5 = -5$$

$$f(1) = (1)^2 - 2(1) - 5 = -6$$

$$f(5) = (5)^2 - 2(5) - 5 = 10$$

Hence, the minimum value of $f(x)$ occurs at $x = 1$, and the maximum value occurs at $x = 5$.

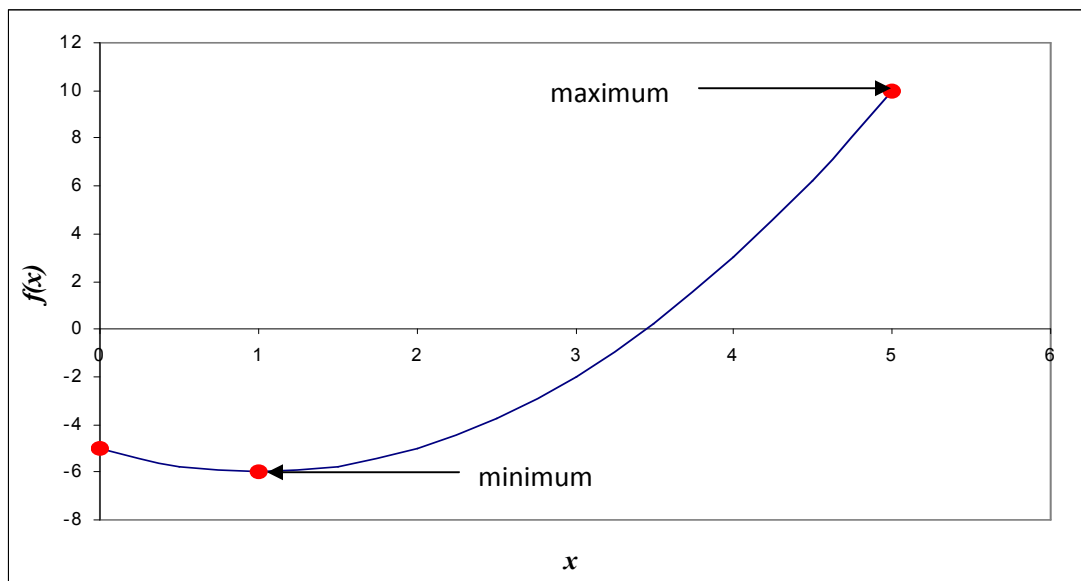


Figure 9 Maximum and minimum values of $f(x) = x^2 - 2x - 5$ over interval $[0,5]$.

Figure 10 shows an example of a function that has no minimum or maximum value in the domain $(0, \infty)$.

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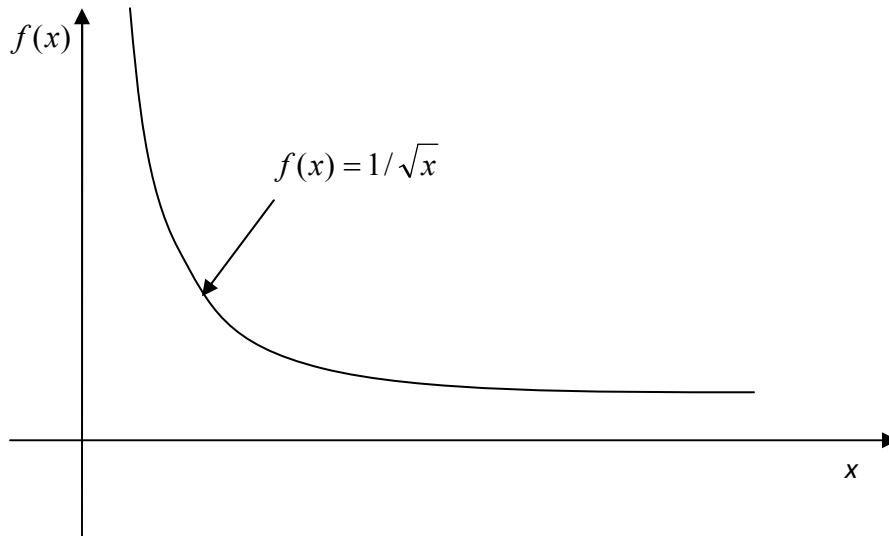


Figure 10 Function that has no maximum or minimum.

Figure 11 shows the maximum of the function occurring at a singular point. The function $f(x)$ has a sharp corner at $x = a$.

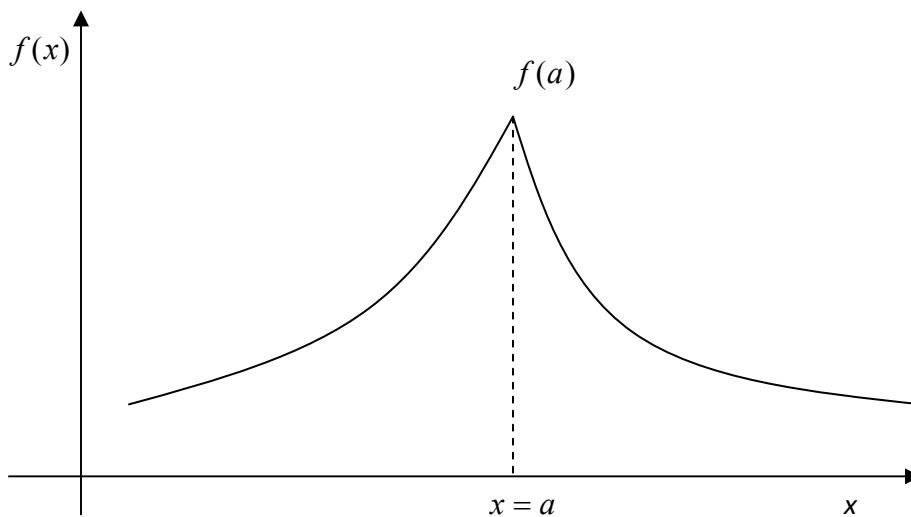


Figure 11 Graph demonstrates the concept of a singular point with discontinuous slope at $x = a$



Example 19

Find the maximum and minimum of $f(x) = 2x$ in the interval $[0,5]$.

Solution

$$f(x) = 2x$$

$$f'(x) = 2$$

$$f'(x) \neq 0 \text{ on } [0,5].$$

So the critical points are $x = 0$ and $x = 5$.

$$f(x) = 2x$$

$$f(0) = 2(0) = 0$$

$$f(5) = 2(5) = 10$$

So the minimum value of $f(x) = 2x$ is at $x = 0$, and the maximum value is at $x = 5$.

The point(s) where the second derivative of a function becomes zero is a way to know whether the critical point found in the first derivative test is a local minimum or maximum. Let $f(x)$ be a function in the interval (c,d) and $f'(a) = 0$.

$f'(a)$ is a local maximum of the function if $f''(a) < 0$.

$f'(a)$ is a local minimum of the function if $f''(a) > 0$.

If $f''(a) = 0$, then the second derivative does not offer any insight into the local maxima or minima.

Example 20

Remember Example 18 where we found $f'(x) = 0$ at $x = 1$ for $f(x) = x^2 - 2x - 5$ in the interval $[0,5]$. Is $x = 1$ a local maxima or minima of the function?

Solution

$$f(x) = x^2 - 2x - 5$$

$$f'(x) = 2x - 2$$

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$$f'(x) = 0 \text{ at } x = 1$$

$$f''(x) = 2$$

$$f''(1) = 2 > 0$$

So the $f(1)$ is the local minimum of the function.

Applications of Derivatives

Below are some examples to show real-life applications of differentiation.

Example 21

A rain gutter cross-section is shown below.

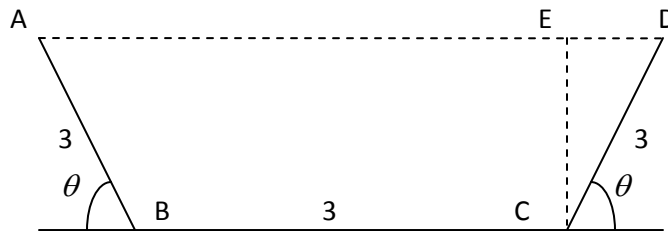


Figure 12 Gutter dimensions for Example 21.

What angle of θ would make the cross-sectional area of ABCD maximum? Note that common sense or intuition may lead us to believe that $\theta = \pi/4$ would maximize the cross-sectional area of ABCD. Question your intuition.

Solution

$$Area = \frac{1}{2}(\overline{BC} + \overline{AD}) \times \overline{CE}$$

$$\overline{CE} = \overline{CD} \sin(\theta)$$

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$$= 3\sin(\theta)$$

$$\overline{BC} = 3$$

$$\overline{AD} = \overline{BC} + \overline{CD}\cos(\theta) + \overline{AB}\cos(\theta)$$

$$\overline{AD} = 3 + 3\cos(\theta) + 3\cos(\theta)$$

$$\overline{AD} = 3 + 6\cos(\theta)$$

$$\begin{aligned} \text{Area} &= \frac{1}{2}(3 + 3 + 6\cos(\theta))(3\sin(\theta)) \\ &= 9\sin(\theta) + 9\sin(\theta)\cos(\theta) \\ &= 9\sin(\theta) + \frac{9}{2}\sin(2\theta) \end{aligned}$$

$$\begin{aligned} \frac{dA}{d\theta} &= 9\cos(\theta) + \frac{9}{2} \times 2\cos(2\theta) \\ &= 9\cos(\theta) + 9\cos(2\theta) \end{aligned}$$

When is

$$\frac{dA}{d\theta} = 0?$$

$$9\cos(\theta) + 9\cos(2\theta) = 0$$

$$\theta = \frac{\pi}{3}$$

The angle at which the area is maximum is $\theta = 60^\circ$.

$$\text{Area}\left(\frac{\pi}{3}\right) = 9\sin\left(\frac{\pi}{3}\right) + \frac{9}{2}\sin\left(2\left(\frac{\pi}{3}\right)\right)$$



$$\begin{aligned}
&= 9\left(\frac{\sqrt{3}}{2}\right) + \frac{9}{2}\left(\frac{\sqrt{3}}{2}\right) \\
&= \frac{27}{4}\sqrt{3}
\end{aligned}$$

For the interval of $\theta = [0, \pi]$, the area at the end points is

$$Area(0) = 0$$

$$Area(\pi) = 0$$

Example 22

A classic example of the application of differentiation is to find the dimensions of a circular cylinder for a specific volume but which uses the least amount of material. Do this classic problem for a volume of $9m^3$.

Solution

The total surface area, A of the cylinder is

$$\begin{aligned}
A &= \text{top surface} + \text{side surface} + \text{bottom surface} \\
&= \pi r^2 + 2\pi r h + \pi r^2 \\
&= 2\pi r^2 + 2\pi r h
\end{aligned}$$

The volume, V of the cylinder is

$$V = \pi r^2 h$$

since

$$V = 9m^3.$$

We can write

$$\begin{aligned}
9 &= \pi r^2 h \\
h &= \frac{9}{\pi r^2}
\end{aligned}$$

This gives the surface area just in terms of r as

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$$\begin{aligned}
 A &= 2\pi r^2 + 2\pi r \left(\frac{9}{\pi r^2} \right) \\
 &= 2\pi r^2 + \frac{18}{r} \\
 &= 2\pi r^2 + 18r^{-1}
 \end{aligned}$$

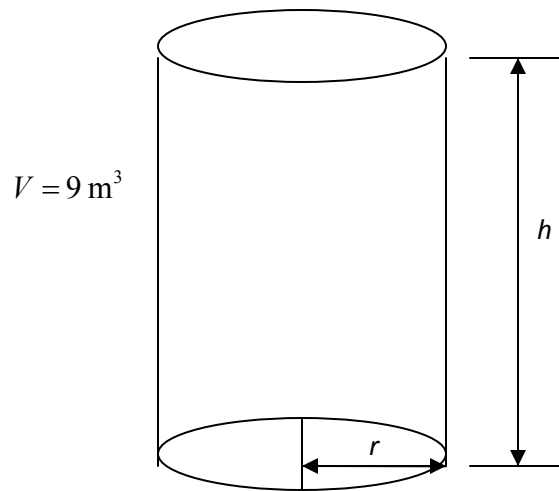


Figure 13 Cylinder drawing for Example 20.

To find the minimum, take the first derivative of A with respect to r as

$$\begin{aligned}
 \frac{dA}{dr} &= 4\pi r + 18(-1)r^{-2} \\
 &= 4\pi r - \frac{18}{r^2}
 \end{aligned}$$

Solving for

$$\frac{dA}{dr} = 0,$$



$$4\pi r - \frac{18}{r^2} = 0$$

$$4\pi r^3 - 18 = 0$$

$$r^3 = \frac{18}{4\pi}$$

$$r = \left(\frac{18}{4\pi}\right)^{\frac{1}{3}}$$
$$= 1.12725 \text{ m}$$

Since

$$h = \frac{9}{\pi r^2},$$

$$h = \frac{9}{\pi(1.12725)^2}$$
$$= 2.25450 \text{ m}$$

But does this value of r correspond to a minimum?

$$\frac{d^2 A}{dr^2} = 4\pi - 18(-2)r^{-3}$$
$$= 4\pi + \frac{36}{r^3}$$
$$= 4\pi + \frac{36}{1.12725}$$
$$= 44.5025$$

This value $\frac{d^2 A}{dr^2} > 0$ for $r = 1.12725 \text{ m}$. As per the second derivative test, $r = 1.12725 \text{ m}$ corresponds to a minimum.

