Mechanical Engineering Example of the Bisection Method
Autar Kaw

Example 1

A trunnion has to be cooled before it is shrink fitted into a steel hub.

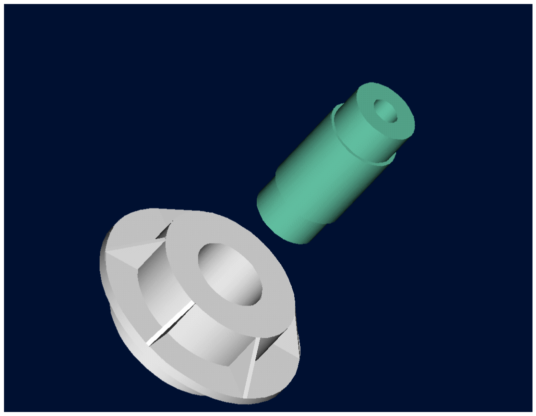

Figure 1 Trunnion to be slid through the hub after contracting.

The equation that gives the temperature T_f to which the trunnion has to be cooled to obtain the desired contraction is given by

$$f(T_f) = -0.50598 \times 10^{-10} T_f^3 + 0.38292 \times 10^{-7} T_f^2 + 0.74363 \times 10^{-4} T_f + 0.88318 \times 10^{-2} = 0$$

Use the bisection method of finding roots of equations to find the temperature T_f to which the trunnion has to be cooled. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

From the designer's records for the previous bridge, the temperature to which the trunnion was cooled was $-108\,°\mathrm{F}$. Hence assuming the temperature to be between $-100\,°\mathrm{F}$ and

$-150\,°\mathrm{F}$, we have

$T_{f,\ell} = -150\,°\text{F}$, $T_{f,u} = -100\,°\text{F}$

Check if the function changes sign between $T_{f,\ell}$ and $T_{f,u}$.

$$f(T_{f,\ell}) = f(-150)$$
$$= -0.50598\times10^{-10}(-150)^3 + 0.38292\times10^{-7}(-150)^2 + 0.74363\times10^{-4}(-150) + 0.88318\times10^{-2}$$
$$= -1.2903\times10^{-3}$$

$$f(T_{f,u}) = f(-100)$$
$$= -0.50598\times10^{-10}(-100)^3 + 0.38292\times10^{-7}(-100)^2 + 0.74363\times10^{-4}(-100) + 0.88318\times10^{-2}$$
$$= 1.8290\times10^{-3}$$

Hence

$$f(T_{f,\ell})f(T_{f,u}) = f(-150)f(-100) = (-1.2903\times10^{-3})(1.8290\times10^{-3}) < 0$$

So there is at least one root between $T_{f,\ell}$ and $T_{f,u}$ that is between -150 and -100.

Iteration 1

The estimate of the root is

$$T_{f,m} = \frac{T_{f,\ell} + T_{f,u}}{2}$$
$$= \frac{-150 + (-100)}{2}$$
$$= -125$$

$$f(T_{f,m}) = f(-125)$$
$$= -0.50598\times10^{-10}(-125)^3 + 0.38292\times10^{-7}(-125)^2 + 0.74363\times10^{-4}(-125) + 0.88318\times10^{-2}$$
$$= 2.3356\times10^{-4}$$

$$f(T_{f,\ell})f(T_{f,m}) = f(-150)f(-125) = (-1.2903\times10^{-3})(2.3356\times10^{-4}) < 0$$

Hence the root is bracketed between $T_{f,\ell}$ and $T_{f,m}$, that is, between -150 and -125.

So, the lower and upper limits of the new bracket are

$$T_{f,\ell} = -150,\ T_{f,u} = -125$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated, as we do not have a previous approximation.

<u>Iteration 2</u>

The estimate of the root is

$$T_{f,m} = \frac{T_{f,\ell} + T_{f,u}}{2}$$

$$= \frac{-150 + (-125)}{2}$$

$$= -137.5$$

$$f(T_{f,m}) = f(-137.5)$$

$$= -0.50598\times10^{-10}(-137.5)^3 + 0.38292\times10^{-7}(-137.5)^2 + 0.74363\times10^{-4}(-137.5) + 0.88318\times10^{-2}$$

$$= -5.3762\times10^{-4}$$

$$f(T_{f,m})f(T_{f,u}) = f(-137.5)f(-125) = (-5.3762\times10^{-4})(2.3356\times10^{-4}) < 0$$

Hence, the root is bracketed between $T_{f,m}$ and $T_{f,u}$, that is, between -125 and -137.5.

So the lower and upper limits of the new bracket are

$$T_{f,\ell} = -137.5, T_{f,u} = -125$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{T_{f,m}^{\text{new}} - T_{f,m}^{\text{old}}}{T_{f,m}^{\text{new}}}\right| \times 100$$

$$= \left|\frac{-137.5 - (-125)}{-137.5}\right| \times 100$$

$$= 9.0909\%$$

None of the significant digits are at least correct in the estimated root of

$$T_{f,m} = -137.5$$

as the absolute relative approximate error is greater that 5%.

Iteration 3

The estimate of the root is

$$T_{f,m} = \frac{T_{f,\ell} + T_{f,u}}{2}$$

$$= \frac{-137.5 + (-125)}{2}$$

$$= -131.25$$

$$f(T_{f,m}) = f(-131.25)$$

$$= -0.50598 \times 10^{-10} (-131.25)^3 + 0.38292 \times 10^{-7} (-131.25)^2$$
$$+ 0.74363 \times 10^{-4} (-131.25) + 0.88318 \times 10^{-2}$$

$$= -1.54303 \times 10^{-4}$$

$$f(T_{f,\ell}) f(T_{f,m}) = f(-125) f(-131.25) = (2.3356 \times 10^{-4})(-1.5430 \times 10^{-4}) < 0$$

Hence, the root is bracketed between $T_{f,\ell}$ and $T_{f,m}$, that is, between -125 and -131.25.

So the lower and upper limits of the new bracket are

$$T_{f,\ell} = -131.25, T_{f,u} = -125$$

The absolute relative approximate error $|\in_a|$ at the ends of Iteration 3 is

$$|\in_a| = \left|\frac{T_{f,m}^{\text{new}} - T_{f,m}^{\text{old}}}{T_{f,m}^{\text{new}}}\right| \times 100$$

$$= \left|\frac{-131.25 - (-137.5)}{-131.25}\right| \times 100$$

$$= 4.7619\%$$

The number of significant digits at least correct is 1.

Seven more iterations were conducted and these iterations are shown in the Table 1 below.

Table 1 Root of $f(x) = 0$ as function of number of iterations for bisection method.

Iteration	$T_{f,\ell}$	$T_{f,u}$	$T_{f,m}$	$\lvert\in_a\rvert\%$	$f(T_{f,m})$
1	−150	−100	−125	---------	2.3356×10^{-4}
2	−150	−125	−137.5	9.0909	-5.3762×10^{-4}
3	−137.5	−125	−131.25	4.7619	-1.5430×10^{-4}
4	−131.25	−125	−128.13	2.4390	
5	−131.25	−128.13	−129.69	1.2048	3.9065×10^{-5}
6	−129.69	−123.13	−128.91	0.60606	-5.7760×10^{-5}
7	−128.91	−123.13	−128.52	0.30395	-9.3826×10^{-6}
8	−128.91	−128.52	−128.71	0.15175	1.4838×10^{-5}
9	−128.91	−128.71	−128.81	0.075815	
10	−128.81	−128.71	−128.76	0.037922	2.7228×10^{-6}
					-3.3305×10^{-6}

					-3.0396×10^{-7}

At the end of the 10^{th} iteration,

$$\left|\in_a\right| = 0.037922\%$$

Hence, the number of significant digits at least correct is given by the largest value of m for which

$$\left|\in_a\right| \le 0.5\times10^{2-m}$$

$$0.037922 \le 0.5\times10^{2-m}$$

$$0.075844 \le 10^{2-m}$$

$$\log(0.075844) \le 2-m$$

$$m \le 2-\log(0.075844) = 3.1201$$

So

$$m = 3$$

The number of significant digits at least correct in the estimated root of -128.76 is 3.