Mechanical Engineering Example of the Gauss-Seidel Method
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Example 1

A trunnion of diameter $12.363"$ has to be cooled from a room temperature of $80\,°\mathrm{F}$ before it is shrink fitted into a steel hub (Figure 1).

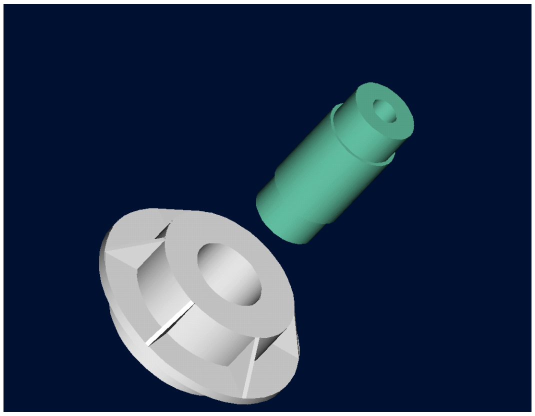

Figure 1 Trunnion to be slid through the hub after contracting.

The equation that gives the diametric contraction ΔD of the trunnion in a dry-ice/alcohol mixture (boiling temperature is $-108\,°\mathrm{F}$) is given by

$$\Delta D = 12.363 \int_{80}^{-108} \alpha(T)\,dT$$

The equation for the thermal expansion coefficient, $\alpha = a_1 + a_2 T + a_3 T^2$, is obtained using regression analysis where the constants of the model are found by solving the following simultaneous linear equations.

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Find the values of a_1, a_2, and a_3 using the Gauss-Seidel method. Use

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

Rewriting the equations gives

$$a_1 = \frac{1.057 \times 10^{-4} - (-2860)a_2 - 7.26 \times 10^5 a_3}{24}$$

$$a_2 = \frac{-1.04162 \times 10^{-2} - (-2860)a_1 - (-1.86472 \times 10^8)a_3}{7.26 \times 10^5}$$

$$a_3 = \frac{2.56799 - 7.26 \times 10^5 a_1 - (-1.86472 \times 10^8)a_2}{5.24357 \times 10^{10}}$$

<u>Iteration #1</u>

Given the initial guess of the solution vector as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we get

$$a_1 = \frac{1.057 \times 10^{-4} - (-2860) \times 0 - 7.26 \times 10^5 \times 0}{24}$$

$$= 4.4042 \times 10^{-6}$$

$$a_2 = \frac{-1.04162 \times 10^{-2} - (-2860) \times 4.4042 \times 10^{-6} - (-1.86472 \times 10^8) \times 0}{7.26 \times 10^5}$$

$$= 3.0024 \times 10^{-9}$$

$$a_3 = \frac{2.56799 - 7.26\times10^5 \times 4.4042\times10^{-6} - (-1.86472\times10^8)\times3.0024\times10^{-9}}{5.24357\times10^{10}}$$

$$= -1.3269\times10^{-12}$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{4.4042\times10^{-6} - 0}{4.4042\times10^{-6}}\right|\times100$$

$$= 100\%$$

$$|\in_a|_2 = \left|\frac{3.0024\times10^{-9} - 0}{3.0024\times10^{-9}}\right|\times100$$

$$= 100\%$$

$$|\in_a|_3 = \left|\frac{-1.3269\times10^{-12} - 0}{-1.3269\times10^{-12}}\right|\times100$$

$$= 100\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.4042\times10^{-6} \\ 3.0024\times10^{-9} \\ -1.3269\times10^{-12} \end{bmatrix}$$

and the maximum absolute relative approximate error is 100%.

<u>Iteration #2</u>

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.4042\times10^{-6} \\ 3.0024\times10^{-9} \\ -1.3269\times10^{-12} \end{bmatrix}$$

Now we get

$$a_1 = \frac{1.057\times10^{-4} - (-2860)\times3.00236\times10^{-9} - 7.26\times10^{5}\times(-1.32692\times10^{-12})}{24}$$

$$= 4.8021\times10^{-6}$$

$$a_2 = \frac{-1.04162\times10^{-2} - (-2860)\times4.8021\times10^{-6} - (-1.86472\times10^{8})\times(-1.3269\times10^{-12})}{7.26\times10^{5}}$$

$$= 4.2291\times10^{-9}$$

$$a_3 = \frac{2.56799 - 7.26\times10^{5}\times4.8021\times10^{-6} - (-1.86472\times10^{8})\times4.2291\times10^{-9}}{5.24357\times10^{10}}$$

$$= -2.4738\times10^{-12}$$

The absolute relative approximate error for each a_i then is

$$\left|\in_a\right|_1 = \left|\frac{4.8021\times10^{-6} - 4.4042\times10^{-6}}{4.8021\times10^{-6}}\right|\times100$$

$$= 8.2864\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.2291\times10^{-9} - 3.0024\times10^{-9}}{4.2291\times10^{-9}}\right|\times100$$

$$= 29.007\%$$

$$\left|\in_a\right|_3 = \left|\frac{-2.4738\times10^{-12} - (-1.3269\times10^{-12})}{-2.4738\times10^{-12}}\right|\times100$$

$$= 46.360\%$$

At the end of the second iteration, the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4.8021\times10^{-6} \\ 4.2291\times10^{-9} \\ -2.4738\times10^{-12} \end{bmatrix}$$

and the maximum absolute relative approximate error is 46.360%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	4.4042×10^{-6}	100	3.0024×10^{-9}	100	-1.3269×10^{-12}	100
2	4.8021×10^{-6}	8.2864	4.2291×10^{-9}	29.0073	-2.4738×10^{-12}	46.3605
3	4.9830×10^{-6}	3.6300	4.6471×10^{-9}	8.9946	-3.4917×10^{-12}	29.1527
4	5.0636×10^{-6}	1.5918	4.7032×10^{-9}	1.1922	-4.4083×10^{-12}	20.7922
5	5.0980×10^{-6}	0.6749	4.6033×10^{-9}	2.1696	-5.2399×10^{-12}	15.8702
6	5.1112×10^{-6}	0.2593	4.4419×10^{-9}	3.6330	-5.9972×10^{-12}	12.6290

After six iterations, the absolute relative approximate errors are decreasing, but they are still high. Allowing for more iterations, the absolute relative approximate errors decrease significantly.

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
75	5.0692×10^{-6}	2.2559×10^{-4}	2.0139×10^{-9}	0.024280	-1.4049×10^{-11}	0.011250
76	5.0691×10^{-6}	2.0630×10^{-4}	2.0135×10^{-9}	0.02221	-1.4051×10^{-11}	0.01029

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.0690\times10^{-6} \\ 2.0087\times10^{-9} \\ -1.4066\times10^{-11} \end{bmatrix}$$