

## Mechanical Engineering Example of the Spline Method of Interpolation Autar Kaw

### Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter  $\Delta D$  of a trunnion shaft by cooling it through a temperature change of  $\Delta T$  is given by

$$\Delta D = D\alpha\Delta T$$

where

$D$  = original diameter (in.)

$\alpha$  = coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from 80°F to -108°F, giving the average temperature as -14°F. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

**Table 1** Thermal expansion coefficient as a function of temperature.

Temperature, $T$ (°F)	Thermal Expansion Coefficient, $\alpha$ (in/in/°F)
80	$6.47 \times 10^{-6}$
0	$6.00 \times 10^{-6}$
-60	$5.58 \times 10^{-6}$
-160	$4.72 \times 10^{-6}$
-260	$3.58 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$

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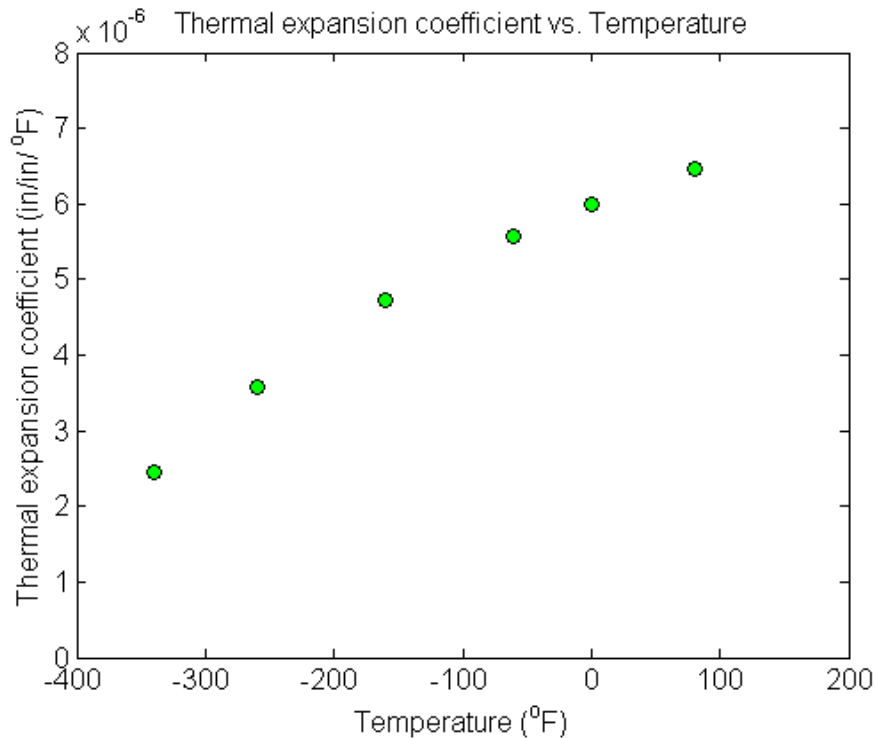
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**Figure 1** Thermal expansion coefficient vs. temperature.

If the coefficient of thermal expansion needs to be calculated at the average temperature of  $-14^{\circ}\text{F}$ , determine the value of the coefficient of thermal expansion at  $T = -14^{\circ}\text{F}$  using linear splines.

### Solution

Since we want to find the coefficient of thermal expansion at  $T = -14^{\circ}\text{F}$  and we are using linear splines, we need to choose the two data points that are closest to  $T = -14^{\circ}\text{F}$  that also bracket  $T = -14^{\circ}\text{F}$  to evaluate it. The two points are  $T_0 = 0$  and  $T_1 = -60^{\circ}\text{F}$ .

Then

$$T_0 = 0, \quad \alpha(T_0) = 6.00 \times 10^{-6}$$

$$T_1 = -60, \quad \alpha(T_1) = 5.58 \times 10^{-6}$$

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gives

$$\begin{aligned}\alpha(T) &= \alpha(T_0) + \frac{\alpha(T_1) - \alpha(T_0)}{T_1 - T_0} (T - T_0) \\ &= 6.00 \times 10^{-6} + \frac{5.58 \times 10^{-6} - 6.00 \times 10^{-6}}{-60 - 0} (T - 0)\end{aligned}$$

Hence

$$\alpha(T) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6} (T - 0), \quad -60 \leq T \leq 0$$

At  $T = -14$ ,

$$\begin{aligned}\alpha(-14) &= 6.00 \times 10^{-6} + 0.007 \times 10^{-6} (-14 - 0) \\ &= 5.902 \times 10^{-6} \text{ in/in/}^\circ\text{F}\end{aligned}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

## Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter  $\Delta D$  of a trunnion shaft by cooling it through a temperature change of  $\Delta T$  is given by

$$\Delta D = D\alpha\Delta T$$

where

$D$  = original diameter (in.)

$\alpha$  = coefficient of thermal expansion at average temperature (in/in/ $^\circ\text{F}$ )

The trunnion is cooled from  $80^\circ\text{F}$  to  $-108^\circ\text{F}$ , giving the average temperature as  $-14^\circ\text{F}$ . The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.



**Table 2** Thermal expansion coefficient as a function of temperature.

Temperature, $T(^{\circ}\text{F})$	Thermal Expansion Coefficient, $\alpha(\text{in/in}/^{\circ}\text{F})$
80	$6.47 \times 10^{-6}$
0	$6.00 \times 10^{-6}$
-60	$5.58 \times 10^{-6}$
-160	$4.72 \times 10^{-6}$
-260	$3.58 \times 10^{-6}$
-340	$2.45 \times 10^{-6}$

- a) Determine the value of the coefficient of thermal expansion at  $T = -14^{\circ}\text{F}$  using quadratic splines. Find the absolute relative approximate error for the second order approximation.
- b) The reduction of the diameter is given more accurately by

$$\Delta D = D \int_{T_r}^{T_f} \alpha dT$$

where  $T_r =$  room temperature ( $^{\circ}\text{F}$ )

$T_f =$  temperature of cooling medium ( $^{\circ}\text{F}$ )

Given

$$T_r = 80^{\circ}\text{F}$$

$$T_f = -108^{\circ}\text{F}$$



find a better estimate. What is the difference between the value found in part (a) and part (b)?

### Solution

a) Since there are six data points, five quadratic splines pass through them.

$$\begin{aligned} \alpha(T) &= a_1T^2 + b_1T + c_1, & -340 \leq T \leq -260 \\ &= a_2T^2 + b_2T + c_2, & -260 \leq T \leq -160 \\ &= a_3T^2 + b_3T + c_3, & -160 \leq T \leq -60 \\ &= a_4T^2 + b_4T + c_4, & -60 \leq T \leq 0 \\ &= a_5T^2 + b_5T + c_5, & 0 \leq T \leq 80 \end{aligned}$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1T^2 + b_1T + c_1$  passes through  $T = -340$  and  $T = -260$ .

$$a_1(-340)^2 + b_1(-340) + c_1 = 2.45 \times 10^{-6} \quad (1)$$

$$a_1(-260)^2 + b_1(-260) + c_1 = 3.58 \times 10^{-6} \quad (2)$$

$a_2T^2 + b_2T + c_2$  passes through  $T = -260$  and  $T = -160$ .

$$a_2(-260)^2 + b_2(-260) + c_2 = 3.58 \times 10^{-6} \quad (3)$$

$$a_2(-160)^2 + b_2(-160) + c_2 = 4.72 \times 10^{-6} \quad (4)$$

$a_3T^2 + b_3T + c_3$  passes through  $T = -160$  and  $T = -60$ .

$$a_3(-160)^2 + b_3(-160) + c_3 = 4.72 \times 10^{-6} \quad (5)$$

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$$a_3(-60)^2 + b_3(-60) + c_3 = 5.58 \times 10^{-6}$$

(6)

$a_4T^2 + b_4T + c_4$  passes through  $T = -60$  and  $T = 0$ .

$$a_4(-60)^2 + b_4(-60) + c_4 = 5.58 \times 10^{-6} \quad (7)$$

$$a_4(0)^2 + b_4(0) + c_4 = 6.00 \times 10^{-6} \quad (8)$$

$a_5T^2 + b_5T + c_5$  passes through  $T = 0$  and  $T = 80$ .

$$a_5(0)^2 + b_5(0) + c_5 = 6.00 \times 10^{-6} \quad (9)$$

$$a_5(80)^2 + b_5(80) + c_5 = 6.47 \times 10^{-6} \quad (10)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At  $T = -260$

$$2a_1(-260) + b_1 - 2a_2(-260) - b_2 = 0 \quad (11)$$

At  $T = -160$

$$2a_2(-160) + b_2 - 2a_3(-160) - b_3 = 0 \quad (12)$$

At  $T = -60$

$$2a_3(-60) + b_3 - 2a_4(-60) - b_4 = 0 \quad (13)$$

At  $T = 0$

$$2a_4(0) + b_4 - 2a_5(0) - b_5 = 0 \quad (14)$$

3. Assuming the first spline  $a_1T^2 + b_1T + c_1$  is linear,

$$a_1 = 0 \quad (15)$$



$$\begin{bmatrix}
 1.156 \times 10^5 & -340 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 67600 & -260 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 25600 & -160 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3600 & -60 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6400 & 80 & 1 & 0 \\
 -520 & 1 & 0 & 520 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -320 & 1 & 0 & 320 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -120 & 1 & 0 & 120 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 a_2 \\
 b_2 \\
 c_2 \\
 a_3 \\
 b_3 \\
 c_3 \\
 a_4 \\
 b_4 \\
 c_4 \\
 a_5 \\
 b_5 \\
 c_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 2.45 \times 10^{-6} \\
 3.58 \times 10^{-6} \\
 3.58 \times 10^{-6} \\
 4.72 \times 10^{-6} \\
 4.72 \times 10^{-6} \\
 5.58 \times 10^{-6} \\
 5.58 \times 10^{-6} \\
 6.00 \times 10^{-6} \\
 6.00 \times 10^{-6} \\
 6.47 \times 10^{-6} \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

Solving the above 15 equations gives the 15 unknowns as

$i$	$a_i$	$b_i$	$c_i$
1	0	$0.014125 \times 10^{-6}$	$7.2525 \times 10^{-6}$
2	$-2.725 \times 10^{-11}$	$-4.5 \times 10^{-11}$	$5.4104 \times 10^{-6}$
3	$-7.5 \times 10^{-13}$	$0.008435 \times 10^{-6}$	$6.0888 \times 10^{-6}$
4	$-2.5417 \times 10^{-11}$	$0.005475 \times 10^{-6}$	$6 \times 10^{-6}$
5	$5 \times 10^{-12}$	$0.005475 \times 10^{-6}$	$6 \times 10^{-6}$

Therefore, the splines are given by

$$\alpha(T) = 0.014125 \times 10^{-6} T + 7.2525 \times 10^{-6}, \quad -340 \leq T \leq -260$$

$$= -2.725 \times 10^{-11} T^2 - 4.5 \times 10^{-11} T + 5.4104 \times 10^{-6}, \quad -260 \leq T \leq -160$$

$$= -7.5 \times 10^{-13} T^2 + 0.008435 \times 10^{-6} T + 6.0888 \times 10^{-6}, \quad -160 \leq T \leq -60$$

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$$= -2.5417 \times 10^{-11} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}, \quad -60 \leq T \leq 0$$

$$= 5 \times 10^{-12} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}, \quad 0 \leq T \leq 80$$

At  $T = -14^\circ\text{F}$

$$\begin{aligned} \alpha(-14) &= -2.5417 \times 10^{-11} (-14)^2 + 0.005475 \times 10^{-6} (-14) + 6 \times 10^{-6} \\ &= 5.9184 \times 10^{-6} \text{ in/in/}^\circ\text{F} \end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the linear and quadratic splines is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{5.9184 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9184 \times 10^{-6}} \right| \times 100 \\ &= 0.27657\% \end{aligned}$$

b) The reduction of the diameter is given more accurately by

$$\Delta D = D \int_{T_r}^{T_f} \alpha dT$$

where

$T_r =$  room temperature ( $^\circ\text{F}$ )

$T_f =$  boiling temperature of liquid nitrogen ( $^\circ\text{F}$ )

Given

$T_r = 80^\circ\text{F}$

$T_f = -108^\circ\text{F}$

To find  $\int_{T_r}^{T_f} \alpha dT$ , we can integrate the quadratic splines with respect to temperature.



$$\begin{aligned}
\int_{T_r}^{T_f} \alpha dT &= \int_{80}^{-108} \alpha(T) dT \\
&= \int_{-60}^{-108} \alpha(T) dT + \int_0^{-60} \alpha(T) dT + \int_{80}^0 \alpha(T) dT \\
&= \int_{-60}^{-108} (-7.5 \times 10^{-13} T^2 + 0.008435 \times 10^{-6} T + 6.0888 \times 10^{-6}) dT \\
&\quad + \int_0^{-60} (-2.5417 \times 10^{-11} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}) dT \\
&\quad + \int_{80}^0 (5 \times 10^{-12} T^2 + 0.005475 \times 10^{-6} T + 6 \times 10^{-6}) dT \\
&= \left[ -7.5 \times 10^{-13} \frac{T^3}{3} + 0.008435 \times 10^{-6} \frac{T^2}{2} + 6.0888 \times 10^{-6} T \right]_{-60}^{-108} \\
&\quad + \left[ -2.5417 \times 10^{-11} \frac{T^3}{3} + 0.005475 \times 10^{-6} \frac{T^2}{2} + 6 \times 10^{-6} T \right]_0^{-60} \\
&\quad + \left[ 5 \times 10^{-12} \frac{T^3}{3} + 0.005475 \times 10^{-6} \frac{T^2}{2} + 6 \times 10^{-6} T \right]_{80}^0 \\
&= [-257.99] \times 10^{-6} + [-348.32] \times 10^{-6} + [-498.373] \times 10^{-6} \\
&= -1104.7 \times 10^{-6}
\end{aligned}$$

$$\int_{T_r}^{T_f} \alpha dT = -1104.7 \times 10^{-6} \text{ in/in}$$

To compare this result with our results from part (a), we take the average coefficient of thermal expansion over this interval, given by:

$$\alpha_{\text{avg}} = \frac{\int_{T_r}^{T_f} \alpha dT}{T_f - T_r}$$



$$\begin{aligned} &= \frac{-1104.7 \times 10^{-6}}{-108 - 80} \\ &= 5.8760 \times 10^{-6} \text{ in/in/}^\circ\text{F} \end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from part (a) and part (b) is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{5.8760 \times 10^{-6} - 5.9184 \times 10^{-6}}{5.8760 \times 10^{-6}} \right| \times 100 \\ &= 0.72178\% \end{aligned}$$

