

## **Romberg Rule of Integration** **Autar Kaw**

*After reading this chapter, you should be able to:*

1. *derive the Romberg rule of integration, and*
2. *use the Romberg rule of integration to solve problems.*

### **What is integration?**

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Romberg rule of approximating integrals of the form

$$I = \int_a^b f(x) dx \tag{1}$$

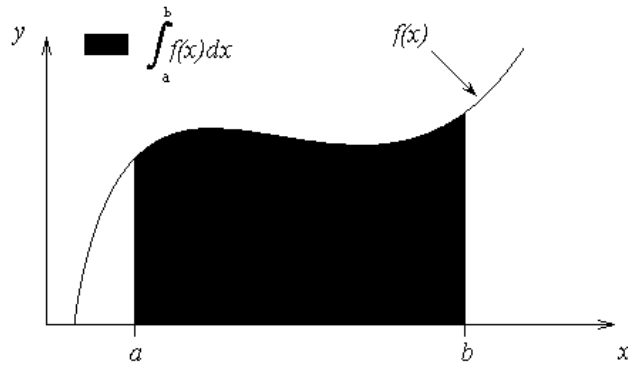
where

$f(x)$  is called the integrand

$a$  = lower limit of integration

$b$  = upper limit of integration





**Figure 1** Integration of a function.

### Error in Multiple-Segment Trapezoidal Rule

The true error obtained when using the multiple segment trapezoidal rule with  $n$  segments to approximate an integral

$$\int_a^b f(x) dx$$

is given by

$$E_i = -\frac{(b-a)^3}{12n^2} \sum_{i=1}^n \frac{f''(\xi_i)}{n} \quad (2)$$

where for each  $i$ ,  $\xi_i$  is a point somewhere in the domain  $[a + (i-1)h, a + ih]$ , and



the term  $\frac{\sum_{i=1}^n f''(\xi_i)}{n}$  can be viewed as an approximate average value of  $f''(x)$  in  $[a, b]$ .

This leads us to say that the true error  $E_t$  in Equation (2) is approximately proportional to

$$E_t \approx \alpha \frac{1}{n^2} \tag{3}$$

for the estimate of  $\int_a^b f(x)dx$  using the  $n$ -segment trapezoidal rule.

Table 1 shows the results obtained for

$$\int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

using the multiple-segment trapezoidal rule.

**Table 1** Values obtained using multiple segment trapezoidal rule for

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt .$$

$n$	Approximate Value	$E_t$	$ \epsilon_t \%$	$ \epsilon_a \%$
1	11868	-807	7.296	---
2	11266	-205	1.854	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594



5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

The true error for the 1-segment trapezoidal rule is  $-807$ , while for the 2-segment rule, the true error is  $-205$ . The true error of  $-205$  is approximately a quarter of  $-807$ . The true error gets approximately quartered as the number of segments is doubled from 1 to 2. The same trend is observed when the number of segments is doubled from 2 to 4 (the true error for 2-segments is  $-205$  and for four segments is  $-51.5$ ). This follows Equation (3).

This information, although interesting, can also be used to get a better approximation of the integral. That is the basis of Richardson's extrapolation formula for integration by the trapezoidal rule.

### Richardson's Extrapolation Formula for Trapezoidal Rule

The true error,  $E_t$ , in the  $n$ -segment trapezoidal rule is estimated as

$$E_t \approx \alpha \frac{1}{n^2}$$

$$E_t \approx \frac{C}{n^2} \tag{4}$$

where  $C$  is an approximate constant of proportionality.

Since

$$E_t = TV - I_n \tag{5}$$

where



$TV$  = true value

$I_n$  = approximate value using  $n$ -segments

Then from Equations (4) and (5),

$$\frac{C}{n^2} \approx TV - I_n \quad (6)$$

If the number of segments is doubled from  $n$  to  $2n$  in the trapezoidal rule,

$$\frac{C}{(2n)^2} \approx TV - I_{2n} \quad (7)$$

Equations (6) and (7) can be solved simultaneously to get

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3} \quad (8)$$

### Example 1

The vertical distance in meters covered by a rocket from  $t = 8$  to  $t = 30$  seconds is given by

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

- Use Romberg's rule to find the distance covered. Use the 2-segment and 4-segment trapezoidal rule results given in Table 1.
- Find the true error for part (a).
- Find the absolute relative true error for part (a).

### Solution

a)  $I_2 = 11266$  m

$$I_4 = 11113$$
 m

Using Richardson's extrapolation formula for the trapezoidal rule, the true value is given by



$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing  $n=2$ ,

$$\begin{aligned} TV &\approx I_4 + \frac{I_4 - I_2}{3} \\ &= 11113 + \frac{11113 - 11266}{3} \\ &= 11062 \text{ m} \end{aligned}$$

b) The exact value of the above integral is

$$\begin{aligned} x &= \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \\ &= 11061 \text{ m} \end{aligned}$$

so the true error

$$\begin{aligned} E_t &= \text{True Value} - \text{Approximate Value} \\ &= 11061 - 11062 \\ &= -1 \text{ m} \end{aligned}$$

c) The absolute relative true error,  $|\epsilon_t|$ , would then be

$$\begin{aligned} |\epsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{11061 - 11062}{11061} \right| \times 100 \\ &= 0.00904\% \end{aligned}$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, and 8 segments. Results are compared with those of the trapezoidal rule.



**Table 2** Values obtained using Richardson's extrapolation formula for the trapezoidal rule for

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt .$$

$n$	Trapezoidal Rule	$ \epsilon_t $ % for Trapezoidal Rule	Richardson's Extrapolation	$ \epsilon_t $ % for Richardson's Extrapolation
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

### Romberg Integration

Romberg integration is the same as Richardson's extrapolation formula as given by Equation (8). However, Romberg used a recursive algorithm for the extrapolation as follows.

The estimate of the true error in the trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^n f''(\xi_i)}{n}$$

Since the segment width,  $h$ , is given by

$$h = \frac{b-a}{n}$$



Equation (2) can be written as

$$E_t = -\frac{h^2(b-a)}{12} \frac{\sum_{i=1}^n f''(\xi_i)}{n} \quad (9)$$

The estimate of true error is given by

$$E_t \approx Ch^2 \quad (10)$$

It can be shown that the exact true error could be written as

$$E_t = A_1h^2 + A_2h^4 + A_3h^6 + \dots \quad (11)$$

and for small  $h$ ,

$$E_t = A_1h^2 + O(h^4) \quad (12)$$

Since we used  $E_t \approx Ch^2$  in the formula (Equation (12)), the result obtained from Equation (10) has an error of  $O(h^4)$  and can be written as

$$\begin{aligned} (I_{2n})_R &= I_{2n} + \frac{I_{2n} - I_n}{3} \\ &= I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1} \end{aligned} \quad (13)$$

where the variable  $TV$  is replaced by  $(I_{2n})_R$  as the value obtained using Richardson's extrapolation formula. Note also that the sign  $\approx$  is replaced by the sign  $=$ .

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Determine another integral value with further halving the step size (doubling the number of segments),

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3} \quad (14)$$

then



$$TV \approx (I_{4n})_R + C\left(\frac{h}{2}\right)^4$$

From Equation (13) and (14),

$$\begin{aligned} TV &\approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} \end{aligned} \quad (15)$$

The above equation now has the error of  $O(h^6)$ . The above procedure can be further improved by using the new values of the estimate of the true value that has the error of  $O(h^6)$  to give an estimate of  $O(h^8)$ .

Based on this procedure, a general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, \quad k \geq 2 \quad (16)$$

The index  $k$  represents the order of extrapolation. For example,  $k = 1$  represents the values obtained from the regular trapezoidal rule,  $k = 2$  represents the values obtained using the true error estimate as  $O(h^2)$ , etc. The index  $j$  represents the more and less accurate estimate of the integral. The value of an integral with a  $j + 1$  index is more accurate than the value of the integral with a  $j$  index.

For  $k = 2, j = 1,$



$$\begin{aligned}
 I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1} \\
 &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}
 \end{aligned}$$

For  $k = 3, j = 1,$

$$\begin{aligned}
 I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1} \\
 &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}
 \end{aligned} \tag{17}$$

## Example 2

The vertical distance in meters covered by a rocket from  $t = 8$  to  $t = 30$  seconds is given by

$$x = \int_8^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Romberg's rule to find the distance covered. Use the 1, 2, 4, and 8-segment trapezoidal rule results as given in Table 1.

## Solution

From Table 1, the needed values from the original the trapezoidal rule are

$$I_{1,1} = 11868$$

$$I_{1,2} = 11266$$

$$I_{1,3} = 11113$$

$$I_{1,4} = 11074$$

where the above four values correspond to using 1, 2, 4 and 8 segment trapezoidal rule, respectively. To get the first order extrapolation values,



$$\begin{aligned}
I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\
&= 11266 + \frac{11266 - 11868}{3} \\
&= 11065
\end{aligned}$$

Similarly

$$\begin{aligned}
I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\
&= 11113 + \frac{11113 - 11266}{3} \\
&= 11062
\end{aligned}$$

$$\begin{aligned}
I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\
&= 11074 + \frac{11074 - 11113}{3} \\
&= 11061
\end{aligned}$$

For the second order extrapolation values,

$$\begin{aligned}
I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\
&= 11062 + \frac{11062 - 11065}{15} \\
&= 11062
\end{aligned}$$

Similarly

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$



$$= 11061 + \frac{11061 - 11062}{15}$$

$$= 11061$$

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 11061 + \frac{11061 - 11062}{63}$$

$$= 11061m$$

Table 3 shows these increasingly correct values in a tree graph.

**Table 3** Improved estimates of the value of an integral using Romberg integration.

			First Order		Second Order		Third Order
1-segment	11868		11065		11062		11061
2-segment	11266		11062		11061		
4-segment	11113		11061				
8-segment	11074						

