

Guass Quadrature Rule of Integration

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After reading this chapter, you should be able to:

1. *derive the Gauss quadrature method for integration and be able to use it to solve problems, and*
2. *use Gauss quadrature method to solve examples of approximate integrals.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the Gauss quadrature rule of approximating integrals of the form

$$I = \int_a^b f(x) dx$$

where

$f(x)$ is called the integrand,

a = lower limit of integration

b = upper limit of integration

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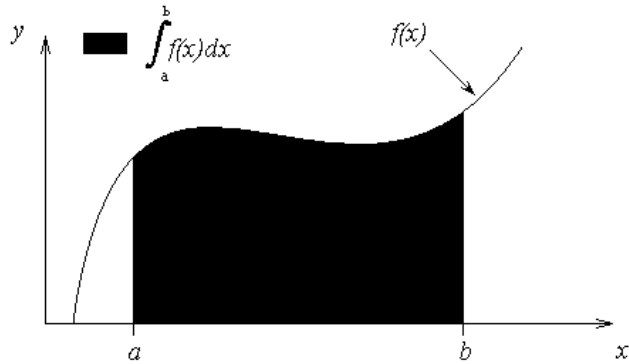


Figure 1 Integration of a function.

Gauss Quadrature Rule

Background:

To derive the trapezoidal rule from the method of undetermined coefficients, we approximated

$$\int_a^b f(x) dx \approx c_1 f(a) + c_2 f(b)$$

(1)

Let the right hand side be exact for integrals of a straight line, that is, for an integrated form of

$$\int_a^b (a_0 + a_1 x) dx$$

So

$$\int_a^b (a_0 + a_1 x) dx = \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$



$$= a_0(b-a) + a_1 \left(\frac{b^2 - a^2}{2} \right)$$

(2)

But from Equation (1), we want

$$\int_a^b (a_0 + a_1x) dx = c_1 f(a) + c_2 f(b)$$

(3)

to give the same result as Equation (2) for $f(x) = a_0 + a_1x$.

$$\begin{aligned} \int_a^b (a_0 + a_1x) dx &= c_1(a_0 + a_1a) + c_2(a_0 + a_1b) \\ &= a_0(c_1 + c_2) + a_1(c_1a + c_2b) \end{aligned}$$

(4)

Hence from Equations (2) and (4),

$$a_0(b-a) + a_1 \left(\frac{b^2 - a^2}{2} \right) = a_0(c_1 + c_2) + a_1(c_1a + c_2b)$$

Since a_0 and a_1 are arbitrary constants for a general straight line

$$c_1 + c_2 = b - a$$

(5a)

$$c_1a + c_2b = \frac{b^2 - a^2}{2}$$

(5b)

Multiplying Equation (5a) by a and subtracting from Equation (5b) gives

$$c_2 = \frac{b-a}{2} \tag{6a}$$

Substituting the above found value of c_2 in Equation (5a) gives



$$c_1 = \frac{b-a}{2} \tag{6b}$$

Therefore

$$\begin{aligned} \int_a^b f(x)dx &\approx c_1 f(a) + c_2 f(b) \\ &= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \end{aligned} \tag{7}$$

Derivation of two-point Gauss quadrature rule

Method 1:

The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as a and b , but as unknowns x_1 and x_2 . So in the two-point Gauss quadrature rule, the integral is approximated as

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &\approx c_1 f(x_1) + c_2 f(x_2) \end{aligned}$$

There are four unknowns x_1 , x_2 , c_1 and c_2 . These are found by assuming that the formula gives exact results for integrating a general third order polynomial,

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Hence

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b (a_0 + a_1x + a_2x^2 + a_3x^3)dx \\ &= \left[a_0x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_a^b \end{aligned}$$



$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right) \quad (8)$$

The formula would then give

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) = c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3) \quad (9)$$

Equating Equations (8) and (9) gives

$$\begin{aligned} & a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right) \\ &= c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3) \\ &= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3) \end{aligned} \quad (10)$$

Since in Equation (10), the constants a_0 , a_1 , a_2 , and a_3 are arbitrary, the coefficients of a_0 , a_1 , a_2 , and a_3 are equal. This gives us four equations as follows.

$$\begin{aligned} b - a &= c_1 + c_2 \\ \frac{b^2 - a^2}{2} &= c_1x_1 + c_2x_2 \\ \frac{b^3 - a^3}{3} &= c_1x_1^2 + c_2x_2^2 \\ \frac{b^4 - a^4}{4} &= c_1x_1^3 + c_2x_2^3 \end{aligned} \quad (11)$$

Without proof (see Example 1 for proof of a related problem), we can find that the above four simultaneous nonlinear equations have only one acceptable solution



$$\begin{aligned}
c_1 &= \frac{b-a}{2} \\
c_2 &= \frac{b-a}{2} \\
x_1 &= \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \\
x_2 &= \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}
\end{aligned} \tag{12}$$

Hence

$$\begin{aligned}
\int_a^b f(x)dx &\approx c_1 f(x_1) + c_2 f(x_2) \\
&= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)
\end{aligned} \tag{13}$$

Method 2:

We can derive the same formula by assuming that the expression gives exact values for the individual integrals of $\int_a^b 1dx$, $\int_a^b xdx$, $\int_a^b x^2dx$, and $\int_a^b x^3dx$. The reason the formula can also be derived using this method is that the linear combination of the above integrands is a general third order polynomial given by $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

These will give four equations as follows

$$\begin{aligned}
\int_a^b 1dx &= b-a = c_1 + c_2 \\
\int_a^b xdx &= \frac{b^2 - a^2}{2} = c_1x_1 + c_2x_2
\end{aligned}$$



$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 x_1^2 + c_2 x_2^2$$

$$\int_a^b x^3 dx = \frac{b^4 - a^4}{4} = c_1 x_1^3 + c_2 x_2^3$$

(14)

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

(15)

Hence

$$\int_a^b f(x) dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) \quad (16)$$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

Higher point Gauss quadrature formulas

For example



$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

(17)

is called the three-point Gauss quadrature rule. The coefficients c_1 , c_2 and c_3 , and the function arguments x_1 , x_2 and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5)dx.$$

General n -point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \dots + c_n f(x_n)$$

(18)

Arguments and weighing factors for n -point Gauss quadrature rules

In handbooks (see Table 1), coefficients and arguments given for n -point Gauss quadrature rule are given for integrals of the form

$$\int_{-1}^1 g(x)dx \approx \sum_{i=1}^n c_i g(x_i)$$

(19)

Table 1 Weighting factors c and function arguments x used in Gauss quadrature formulas

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$

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3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$
5	$c_1 = 0.236926885$	$x_1 = -0.906179846$
	$c_2 = 0.478628670$	$x_2 = -0.538469310$
	$c_3 = 0.568888889$	$x_3 = 0.000000000$
	$c_4 = 0.478628670$	$x_4 = 0.538469310$
	$c_5 = 0.236926885$	$x_5 = 0.906179846$
6	$c_1 = 0.171324492$	$x_1 = -0.932469514$
	$c_2 = 0.360761573$	$x_2 = -0.661209386$
	$c_3 = 0.467913935$	$x_3 = -0.238619186$
	$c_4 = 0.467913935$	$x_4 = 0.238619186$

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	$c_5 = 0.360761573$	$x_5 = 0.661209386$
	$c_6 = 0.171324492$	$x_6 = 0.932469514$

So if the table is given for $\int_{-1}^1 g(x)dx$ integrals, how does one solve $\int_a^b f(x)dx$?

The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$. Let

$$x = mt + c$$

(20)

If $x = a$, then $t = -1$

If $x = b$, then $t = +1$

such that

$$a = m(-1) + c$$

$$b = m(1) + c$$

(21)

Solving the two Equations (21) simultaneously gives

$$m = \frac{b - a}{2}$$

$$c = \frac{b + a}{2}$$

(22)

Hence

$$x = \frac{b - a}{2}t + \frac{b + a}{2}$$

$$dx = \frac{b - a}{2}dt$$



Substituting our values of x and dx into the integral gives us

$$\int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2} dx$$

(23)

Example 1

For an integral $\int_{-1}^1 f(x)dx$, show that the two-point Gauss quadrature rule approximates to

$$\int_{-1}^1 f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

where

$$c_1 = 1$$

$$c_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Solution

Assuming the formula

$$\int_{-1}^1 f(x)dx = c_1 f(x_1) + c_2 f(x_2) \tag{E1.1}$$

gives exact values for integrals $\int_{-1}^1 1dx$, $\int_{-1}^1 xdx$, $\int_{-1}^1 x^2 dx$, and $\int_{-1}^1 x^3 dx$. Then



$$\int_{-1}^1 1 dx = 2 = c_1 + c_2$$

(E1.2)

$$\int_{-1}^1 x dx = 0 = c_1 x_1 + c_2 x_2$$

(E1.3)

$$\int_{-1}^1 x^2 dx = \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2$$

(E1.4)

$$\int_{-1}^1 x^3 dx = 0 = c_1 x_1^3 + c_2 x_2^3$$

(E1.5)

Multiplying Equation (E1.3) by x_1^2 and subtracting from Equation (E1.5) gives

$$c_2 x_2 (x_1^2 - x_2^2) = 0$$

(E1.6)

The solution to the above equation is

$$c_2 = 0, \text{ or/and}$$

$$x_2 = 0, \text{ or/and}$$

$$x_1 = x_2, \text{ or/and}$$

$$x_1 = -x_2.$$

- I. $c_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 = 2$, $c_1 x_1 = 0$, $c_1 x_1^2 = \frac{2}{3}$, and $c_1 x_1^3 = 0$. But since $c_1 = 2$, then $x_1 = 0$ from $c_1 x_1 = 0$, but $x_1 = 0$ conflicts with $c_1 x_1^2 = \frac{2}{3}$.



II. $x_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1 x_1 = 0$, $c_1 x_1^2 = \frac{2}{3}$, and $c_1 x_1^3 = 0$. Since $c_1 x_1 = 0$, then c_1 or x_1 has to be zero but this violates $c_1 x_1^2 = \frac{2}{3} \neq 0$.

III. $x_1 = x_2$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1 x_1 + c_2 x_1 = 0$, $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3}$, and $c_1 x_1^3 + c_2 x_1^3 = 0$. If $x_1 \neq 0$, then $c_1 x_1 + c_2 x_1 = 0$ gives $c_1 + c_2 = 0$ and that violates $c_1 + c_2 = 2$. If $x_1 = 0$, then that violates $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3} \neq 0$.

That leaves the solution of $x_1 = -x_2$ as the only possible acceptable solution and in fact, it does not have violations (see it for yourself)

$$x_1 = -x_2$$

(E1.7)

Substituting (E1.7) in Equation (E1.3) gives

$$c_1 = c_2$$

(E1.8)

From Equations (E1.2) and (E1.8),

$$c_1 = c_2 = 1$$

(E1.9)

Equations (E1.4) and (E1.9) gives

$$x_1^2 + x_2^2 = \frac{2}{3}$$

(E1.10)

Since Equation (E1.7) requires that the two results be of opposite sign, we get

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$



Hence

$$\int_{-1}^1 f(x)dx = c_1 f(x_1) + c_2 f(x_2) \quad (\text{E1.11})$$
$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Example 2

For an integral $\int_a^b f(x)dx$, derive the one-point Gauss quadrature rule.

Solution

The one-point Gauss quadrature rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1)$$

(E2.1)

Assuming the formula gives exact values for integrals $\int_{-1}^1 1dx$, and $\int_{-1}^1 xdx$

$$\int_a^b 1dx = b - a = c_1$$
$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1$$

(E2.2)

Since $c_1 = b - a$, the other equation becomes

$$(b - a)x_1 = \frac{b^2 - a^2}{2}$$



$$x_1 = \frac{b+a}{2}$$

(E2.3)

Therefore, one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{b+a}{2}\right)$$

(E2.4)

Example 3

What would be the formula for

$$\int_a^b f(x)dx = c_1f(a) + c_2f(b)$$

if you want the above formula to give you exact values of $\int_a^b (a_0x + b_0x^2)dx$, that is, a linear combination of x and x^2 .

Solution

If the formula is exact for a linear combination of x and x^2 , then

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1a + c_2b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1a^2 + c_2b^2$$

(E3.1)

Solving the two Equations (E3.1) simultaneously gives

$$\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{b^2 - a^2}{2} \\ \frac{b^3 - a^3}{3} \end{bmatrix}$$

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$$c_1 = -\frac{1 - ab - b^2 + 2a^2}{6a}$$

$$c_2 = -\frac{1}{6} \frac{a^2 + ab - 2b^2}{b}$$

(E3.2)

So

$$\int_a^b f(x) dx = -\frac{1 - ab - b^2 + 2a^2}{6a} f(a) - \frac{1}{6} \frac{a^2 + ab - 2b^2}{b} f(b) \quad (\text{E3.3})$$

Let us see if the formula works.

Evaluate $\int_2^5 (2x^2 - 3x) dx$ using Equation(E3.3)

$$\begin{aligned} \int_2^5 (2x^2 - 3x) dx &\approx c_1 f(a) + c_2 f(b) \\ &= -\frac{1 - (2)(5) - 5^2 + 2(2)^2}{6 \cdot 2} [2(2)^2 - 3(2)] - \frac{1}{6} \frac{2^2 + 2(5) - 2(5)^2}{5} [2(5)^2 - 3(5)] \\ &= 46.5 \end{aligned}$$

The exact value of $\int_2^5 (2x^2 - 3x) dx$ is given by

$$\begin{aligned} \int_2^5 (2x^2 - 3x) dx &= \left[\frac{2x^3}{3} - \frac{3x^2}{2} \right]_2^5 \\ &= 46.5 \end{aligned}$$

Any surprises?

Now evaluate $\int_2^5 3 dx$ using Equation (E3.3)



$$\int_2^5 3dx \approx c_1 f(a) + c_2 f(b)$$

$$= -\frac{1}{6} \frac{1 - 2(5) - 5^2 + 2(2)^2}{2} (3) - \frac{1}{6} \frac{2^2 + 2(5) - 2(5)^2}{5} (3)$$

$$= 10.35$$

The exact value of $\int_2^5 3dx$ is given by

$$\int_2^5 3dx = [3x]_2^5$$

$$= 9$$

Because the formula will only give exact values for linear combinations of x and x^2 , it does not work exactly even for a simple integral of $\int_2^5 3dx$.

Do you see now why we choose $a_0 + a_1x$ as the integrand for which the formula

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

gives us exact values?

Example 4

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Also, find the absolute relative true error.



Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation(23) gives

$$\begin{aligned}\int_8^{30} f(t)dt &= \frac{30-8}{2} \int_{-1}^1 f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx \\ &= 11 \int_{-1}^1 f(11x+19)dx\end{aligned}$$

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$$c_1 = 1.000000000.$$

$$x_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$x_2 = 0.577350269$$

Now we can use the Gauss quadrature formula

$$\begin{aligned}11 \int_{-1}^1 f(11x+19)dx &\approx 11[c_1 f(11x_1+19) + c_2 f(11x_2+19)] \\ &= 11[f(11(-0.5773503)+19) + f(11(0.5773503)+19)] \\ &= 11[f(12.64915) + f(25.35085)] \\ &= 11[(296.8317) + (708.4811)] \\ &= 11058.44 \text{ m}\end{aligned}$$

since

$$\begin{aligned}f(12.64915) &= 2000 \ln \left[\frac{140000}{140000 - 2100(12.64915)} \right] - 9.8(12.64915) \\ &= 296.8317\end{aligned}$$



$$f(25.35085) = 2000 \ln \left[\frac{140000}{140000 - 2100(25.35085)} \right] - 9.8(25.35085)$$

$$= 708.4811$$

The absolute relative true error, $|\epsilon_t|$, is (True value = 11061.34 m)

$$|\epsilon_t| = \left| \frac{11061.34 - 11058.44}{11061.34} \right| \times 100$$

$$= 0.0262\%$$

Example 5

Use three-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation (23) gives

$$\int_8^{30} f(t) dt = \frac{30-8}{2} \int_{-1}^1 f \left(\frac{30-8}{2}x + \frac{30+8}{2} \right) dx$$

$$= 11 \int_{-1}^1 f(11x+19) dx$$

The weighting factors and function argument values are

$$c_1 = 0.555555556$$

$$x_1 = -0.774596669$$



$$c_2 = 0.888888889$$

$$x_2 = 0.000000000$$

$$c_3 = 0.555555556$$

$$x_3 = 0.774596669$$

and the formula is

$$\begin{aligned} 11 \int_{-1}^1 f(11x+19) dx &\approx 11 [c_1 f(11x_1+19) + c_2 f(11x_2+19) + c_3 f(11x_3+19)] \\ &= 11 \left[0.5555556 f(11(-.7745967)+19) + 0.8888889 f(11(0.0000000)+19) \right. \\ &\quad \left. + 0.5555556 f(11(0.7745967)+19) \right] \\ &= 11 [0.55556 f(10.47944) + 0.88889 f(19.00000) + 0.55556 f(27.52056)] \\ &= 11 [0.55556 \times 239.3327 + 0.88889 \times 484.7455 + 0.55556 \times 795.1069] \\ &= 11061.31 \text{ m} \end{aligned}$$

since

$$\begin{aligned} f(10.47944) &= 2000 \ln \left[\frac{140000}{140000 - 2100(10.47944)} \right] - 9.8(10.47944) \\ &= 239.3327 \end{aligned}$$

$$\begin{aligned} f(19.00000) &= 2000 \ln \left[\frac{140000}{140000 - 2100(19.00000)} \right] - 9.8(19.00000) \\ &= 484.7455 \end{aligned}$$

$$\begin{aligned} f(27.52056) &= 2000 \ln \left[\frac{140000}{140000 - 2100(27.52056)} \right] - 9.8(27.52056) \\ &= 795.1069 \end{aligned}$$

The absolute relative true error, $|\epsilon_t|$, is (True value = 11061.34 m)

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$$|\epsilon_t| = \left| \frac{11061.34 - 11061.31}{11061.34} \right| \times 100$$
$$= 0.0003\%$$

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