Computer Engineering Example of the Gauss Quadrature Rule

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Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_{0}^{100} f(x)dx$$

Where,

$$\begin{aligned} f(x) &= 0,\ 0 < x < 30 \\ &= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,\ 30 \le x \le 172 \\ &= 0,\ 172 < x < 200 \end{aligned}$$

Use two-point Gauss Quadrature Rule to find the value of the integral.

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[0, 100]$ to $[-1, 1]$ using

$$\int_{a}^{b} f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

gives

$$\int_{0}^{100} f(x)dx = \frac{100-0}{2}\int_{-1}^{1} f\left(\frac{100-0}{2}x + \frac{100+0}{2}\right)dx$$

$$= 50\int_{-1}^{1} f(50x+50)dx$$

Next, get weighting factors and function argument values for the two point rule,

$$c_1 = \ 1.0000$$

$$x_1 = -0.57735$$

$$c_2 = 1.0000$$

$$x_2 = 0.57735$$

Now we can use the Gauss Quadrature formula

$$50\int_{-1}^{1} f(50x+50)dx \approx 50\left[c_1 f(50x_1+50)+c_2 f(50x_2+50)\right]$$

$$\approx 50\left[f(50(-0.57735)+50)+f(50(0.57735)+50)\right]$$

$$\approx 50\left[f(21.132)+f(78.868)\right]$$

$$\approx 50\left[(0)+(0.10492)\right]$$

$$\approx 5.2460$$

since

$$f(21.132)=0$$

$$f(78.868) = -9.1688\times10^{-6}\times(78.868)^3+2.7961\times10^{-3}\times(78.868)^2 - 2.8487\times10^{-1}\times(78.868)+9.6778$$

$$= 0.10492$$

The absolute relative true error, $|\in_t|$, is (Exact value = 60.793)

$$|\in_t| = \left|\frac{60.793-5.2460}{60.793}\right|\times 100\,\%$$

$$= 91.371\,\%$$

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_0^{100} f(x)dx$$

where

$$\begin{aligned} f(x) &= 0,\ 0 < x < 30 \\ &= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,\ \ 30 \le x \le 172 \\ &= 0,\ \ 172 < x < 200 \end{aligned}$$

Use three-point Gauss Quadrature to find the value of the integral.

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[0, 100]$ to $[-1, 1]$ using

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

gives

$$\int_0^{100} f(x)dx = \frac{100-0}{2}\int_{-1}^{1} f\left(\frac{100-0}{2}x + \frac{100+0}{2}\right)dx$$

$$= 50\int_{-1}^{1} f(50x+50)dx$$

The weighting factors and function argument values are

$$c_1 = \ \ 0.55556$$

$$x_1 = -0.77460$$

$$c_2 = 0.88889$$

$$x_2 = 0.0000$$

$$c_3 = 0.55556$$

$$x_3 = 0.77460$$

and the formula is

$$50\int_{-1}^{1} f(50x+50)dx \approx 50[c_1 f(50x_1+50) + c_2 f(50x_2+50) + c_3 f(50x_3+50)]$$

$$\approx 50\begin{bmatrix} 0.55556 f(50(-0.77460)+50) \\ +0.88889 f(50(0.0000)+50) \\ +0.55556 f(50(0.77460)+50) \end{bmatrix}$$

$$\approx 50[0.55556 f(11.270) + 0.88889 f(50) + 0.55556 f(88.729)]$$

$$\approx 50[0.55556(0) + 0.88889(1.2785) + 0.55556(0.0099462)]$$

$$\approx 57.096$$

since

$$f(11.270) = 0$$

$$f(50) = -9.1688\times10^{-6}\times(50)^3 + 2.7961\times10^{-3}\times(50)^2 - 2.8487\times10^{-1}\times(50) + 9.6778$$

$$= 1.2785$$

$$f(88.730) = -9.1688\times10^{-6}\times(88.729)^3 + 2.7961\times10^{-3}\times(88.729)^2$$
$$-2.8487\times10^{-1}\times(88.729) + 9.6778$$

$$= 0.0099462$$

The absolute relative true error, $|\in_t|$, is (Exact value = 60.793)

$$|\in_t| = \left|\frac{60.793 - 57.096}{60.793}\right| \times 100\,\%$$

$$= 6.0802\,\%$$