Permutation group

In mathematics, a permutation group is a group \( G \) whose elements are permutations of a given set \( M \), and whose group operation is the composition of permutations in \( G \) (which are thought of as bijective functions from the set \( M \) to itself); the relationship is often written as \((G,M)\). Note that the group of all permutations of a set is the symmetric group; the term permutation group is usually restricted to mean a subgroup of the symmetric group. The symmetric group of \( n \) elements is denoted by \( S_n \); if \( M \) is any finite or infinite set, then the group of all permutations of \( M \) is often written as \( \text{Sym}(M) \).

The application of a permutation group to the elements being permuted is called its group action; it has applications in both the study of symmetries, combinatorics and many other branches of mathematics, physics and chemistry.

Closure properties

As a subgroup of a symmetric group, all that is necessary for a permutation group to satisfy the group axioms is that it contain the identity permutation, the inverse permutation of each permutation it contains, and be closed under composition of its permutations. A general property of finite groups implies that a finite subset of a symmetric group is again a group if and only if it is closed under the group operation.

Examples

Permutations are often written in cyclic form\(^1\) so that given the set \( M = \{1,2,3,4\} \), a permutation \( g \) of \( M \) with \( g(1) = 2, g(2) = 4, g(4) = 1 \) and \( g(3) = 3 \) will be written as \( (1,4)(3) \), or more commonly, \( (1,2,4) \) since 3 is left unchanged; if the objects are denoted by a single letter or digit, commas are also dispensed with, and we have a notation such as \( (1\ 2\ 4) \).

Consider the following set \( G \) of permutations of the set \( M = \{1,2,3,4\} \):

- \( e = (1)(2)(3)(4) = (1) \)\(^2\) \[^3\]
  - This is the identity, the trivial permutation which fixes each element.
- \( a = (1\ 2)(3)(4) = (1\ 2) \)
  - This permutation interchanges 1 and 2, and fixes 3 and 4.
- \( b = (1)(2)(3\ 4) = (3\ 4) \)
  - Like the previous one, but exchanging 3 and 4, and fixing the others.
- \( ab = (1\ 2)(3\ 4) \)
  - This permutation, which is the composition of the previous two, exchanges simultaneously 1 with 2, and 3 with 4.

\( G \) forms a group, since \( aa = bb = e, ba = ab, \) and \( baba = e \). So \((G,M)\) forms a permutation group.

The Rubik's Cube puzzle is another example of a permutation group. The underlying set being permuted is the coloured subcubes of the whole cube. Each of the rotations of the faces of the cube is a permutation of the positions and orientations of the subcubes. Taken together, the rotations form a generating set, which in turn generates a group by composition of these rotations. The axioms of a group are easily seen to be satisfied; to invert any sequence of rotations, simply perform their opposites, in reverse order.\(^3\)

The group of permutations on the Rubik's Cube does not form a complete symmetric group of the 20 corner and face cubelets; there are some final cube positions which cannot be achieved through the legal manipulations of the cube.

More generally, every group \( G \) is isomorphic to a permutation group by virtue of its regular action on \( G \) as a set; this is the content of Cayley's theorem.
**Isomorphisms**

If $G$ and $H$ are two permutation groups on the same set $X$, then we say that $G$ and $H$ are **isomorphic as permutation groups** if there exists a bijective map $f : X \to X$ such that $r \mapsto f^{-1} \circ r \circ f$ defines a bijective map between $G$ and $H$; in other words, if for each element $g$ in $G$, there is a unique $h_g$ in $H$ such that for all $x$ in $X$, $(g \circ f)(x) = (f \circ h_g)(x)$. This is equivalent to $G$ and $H$ being conjugate as subgroups of $\text{Sym}(X)$. In this case, $G$ and $H$ are also isomorphic as groups.

Notice that different permutation groups may well be isomorphic as abstract groups, but not as permutation groups. For instance, the permutation group on $\{1,2,3,4\}$ described above is isomorphic as a group (but not as a permutation group) to $\{(1)(2)(3)(4), (12)(34), (13)(24), (14)(23)\}$. Both are isomorphic as groups to the Klein group $V_4$.

**Transpositions, simple transpositions, inversions and sorting**

A 2-cycle is known as a transposition. A **simple transposition** in $S_n$ is a 2-cycle of the form $(i \ i+1)$.

For a permutation $p$ in $S_n$, a pair $(i, j) \in I_n$ is a **permutation inversion**, if when $i < j$, we have $p(i) > p(j)$.[4]

Every permutation can be written as a product of simple transpositions; furthermore, the number of simple transpositions one can write a permutation $p$ in $S_n$ can be the number of inversions of $p$ and if the number of inversions in $p$ is odd or even the number of transpositions in $p$ will also be odd or even corresponding to the oddness of $p$.

**Notes**

[1] e.g. during cycle index computations
[2] This is just a notation that is often used

**References**
