Solvable group

In mathematics, more specifically in the field of group theory, a solvable group (or soluble group) is a group that can be constructed from abelian groups using extensions. That is, a solvable group is a group whose derived series terminates in the trivial subgroup.

Historically, the word "solvable" arose from Galois theory and the proof of the general unsolvability of quintic equation. Specifically, a polynomial equation is solvable by radicals if and only if the corresponding Galois group is solvable.

Definition

A group $G$ is called solvable if it has a subnormal series whose factor groups are all abelian, that is, if there are subgroups $\{1\} = G_0 \leq G_1 \leq \cdots \leq G_k = G$ such that $G_{j-1}$ is normal in $G_j$, and $G_j/G_{j-1}$ is an abelian group, for $j = 1, 2, \ldots, k$.

Or equivalently, if its derived series, the descending normal series

$G \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \cdots$,

where every subgroup is the commutator subgroup of the previous one, eventually reaches the trivial subgroup $\{1\}$ of $G$. These two definitions are equivalent, since for every group $H$ and every normal subgroup $N$ of $H$, the quotient $H/N$ is abelian if and only if $N$ includes $H^{(1)}$. The least $n$ such that $G^{(n)} = \{1\}$ is called the derived length of the solvable group $G$.

For finite groups, an equivalent definition is that a solvable group is a group with a composition series all of whose factors are cyclic groups of prime order. This is equivalent because a finite abelian group has finite composition length, and every finite simple abelian group is cyclic of prime order. The Jordan–Hölder theorem guarantees that if one composition series has this property, then all composition series will have this property as well. For the Galois group of a polynomial, these cyclic groups correspond to $n$th roots (radicals) over some field. The equivalence does not necessarily hold for infinite groups: for example, since every nontrivial subgroup of the group $Z$ of integers under addition is isomorphic to $Z$ itself, it has no composition series, but the normal series $\{0, Z\}$, with its only factor group isomorphic to $Z$, proves that it is in fact solvable.

In keeping with George Pólya's dictum that "if there's a problem you can't figure out, there's a simpler problem you can figure out", solvable groups are often useful for reducing a conjecture about a complicated group into a conjecture about a series of groups with simple structure: abelian groups (and in the finite case, cyclic groups of prime order).

Examples

All abelian groups are solvable - the quotient $A/B$ will always be abelian if $A$ is abelian. But non-abelian groups may or may not be solvable.

More generally, all nilpotent groups are solvable. In particular, finite $p$-groups are solvable, as all finite $p$-groups are nilpotent.

A small example of a solvable, non-nilpotent group is the symmetric group $S_3$. In fact, as the smallest simple non-abelian group is $A_5$, (the alternating group of degree 5) it follows that every group with order less than 60 is solvable.

The group $S_4$ is not solvable — it has a composition series $\{E, A_4, S_4\}$ (and the Jordan–Hölder theorem states that every other composition series is equivalent to that one), giving factor groups isomorphic to $A_4$ and $C_2^2$, and $A_4$ is not abelian. Generalizing this argument, coupled with the fact that $A_n$ is a normal, maximal, non-abelian simple subgroup of $S_n$ for $n > 4$, we see that $S_n$ is not solvable for $n > 4$, a key step in the proof that for every $n > 4$ there are
solvable group of degree $n$ which are not solvable by radicals.

The celebrated Feit–Thompson theorem states that every finite group of odd order is solvable. In particular this implies that if a finite group is simple, it is either a prime cyclic or of even order.

Any finite group whose every $p$-Sylow subgroups is cyclic is a semidirect product of two cyclic groups, in particular solvable. Such groups are called $Z$-groups.

### Properties

Solvability is closed under a number of operations.

- If $G$ is solvable, and there is a homomorphism from $G$ onto $H$, then $H$ is solvable; equivalently (by the first isomorphism theorem), if $G$ is solvable, and $N$ is a normal subgroup of $G$, then $G/N$ is solvable.
- The previous property can be expanded into the following property: $G$ is solvable if and only if both $N$ and $G/N$ are solvable.
- If $G$ is solvable, and $H$ is a subgroup of $G$, then $H$ is solvable.
- If $G$ and $H$ are solvable, the direct product $G \times H$ is solvable.

Solvability is closed under group extension:

- If $H$ and $G/H$ are solvable, then so is $G$; in particular, if $N$ and $H$ are solvable, their semidirect product is also solvable.

It is also closed under wreath product:

- If $G$ and $H$ are solvable, and $X$ is a $G$-set, then the wreath product of $G$ and $H$ with respect to $X$ is also solvable.

For any positive integer $N$, the solvable groups of derived length at most $N$ form a subvariety of the variety of groups, as they are closed under the taking of homomorphic images, subalgebras, and (direct) products. The direct product of a sequence of solvable groups with unbounded derived length is not solvable, so the class of all solvable groups is not a variety.

### Burnside's theorem

Burnside's theorem states that if $G$ is a finite group of order $p^aq^b$

where $p$ and $q$ are prime numbers, and $a$ and $b$ are non-negative integers, then $G$ is solvable.

### Related concepts

#### Supersolvable groups

As a strengthening of solvability, a group $G$ is called supersolvable (or supersoluble) if it has an invariant normal series whose factors are all cyclic. Since a normal series has finite length by definition, uncountable groups are not supersolvable. In fact, all supersolvable groups are finitely generated, and an abelian group is supersolvable if and only if it is finitely generated. The alternating group $A_4$ is an example of a finite solvable group that is not supersolvable.

If we restrict ourselves to finitely generated groups, we can consider the following arrangement of classes of groups:

- cyclic $<$ abelian $<$ nilpotent $<$ supersolvable $<$ polycyclic $<$ solvable $<$ finitely generated group.
Virtually solvable groups
A group $G$ is called **virtually solvable** if it has a solvable subgroup of finite index. This is similar to virtually abelian. Clearly all solvable groups are virtually solvable, since one can just choose the group itself, which has index 1.

Hypoabelian
A solvable group is one whose derived series reaches the trivial subgroup at a *finite* stage. For an infinite group, the finite derived series may not stabilize, but the transfinite derived series always stabilizes. A group whose transfinite derived series reaches the trivial group is called a **hypoabelian group**, and every solvable group is a hypoabelian group. The first ordinal $\alpha$ such that $G^{(\alpha)} = G^{(\alpha+1)}$ is called the (transfinite) derived length of the group $G$, and it has been shown that every ordinal is the derived length of some group (Malcev 1949).

References

External links
- Sequence A056866[^1] in the OEIS - orders of non-solvable finite groups.

References
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