Chapter 9

Exponential Growth and Decay: Differential Equations

9.1 Observations about the exponential function

In a previous chapter we made an observation about a special property of the function

\[ y = f(x) = e^x \]

namely, that

\[ \frac{dy}{dx} = e^x = y \]

so that this function satisfies the relationship

\[ \frac{dy}{dx} = y. \]

We call this a differential equation because it connects one (or more) derivatives of a function with the function itself.

In this chapter we will study the implications of the above observation. Since most of the applications that we examine will be time-dependent processes, we will here use \( t \) (for time) as the independent variable.

Then we can make the following observations:

1. Let \( y \) be the function of time:

\[ y = f(t) = e^t \]

Then

\[ \frac{dy}{dt} = e^t = y \]

With this slight change of notation, we see that the function \( y = e^t \) satisfies the differential equation

\[ \frac{dy}{dt} = y. \]
2. Now consider
\[ y = e^{kt}. \]

Then, using the chain rule, and setting \( u = kt \), and \( y = e^u \) we find that
\[
\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dt} = e^u \cdot k = ke^{kt} = ky.
\]

So we see that the function \( y = e^{kt} \) satisfies the differential equation
\[
\frac{dy}{dt} = ky.
\]

3. If instead we had the function
\[ y = e^{-kt} \]

we could similarly show that the differential equation it satisfies is
\[
\frac{dy}{dt} = -ky.
\]

4. Now suppose we had a constant in front, e.g. we were interested in the function
\[ y = 5e^{kt}. \]

Then, by simple differentiation and rearrangement we have
\[
\frac{dy}{dt} = 5 \frac{d}{dt} e^{kt} = 5(ke^{kt}) = k(5e^{kt}) = ky.
\]

So we see that this function with the constant in front also satisfies the differential equation
\[
\frac{dy}{dt} = ky.
\]

5. The conclusion we reached in the previous step did not depend at all on the constant out front. Indeed, if we had started with a function of the form
\[ y = Ce^{kt} \]

where \( C \) is any constant, we would still have a function that satisfies the same differential equation.

6. While we will not prove this here, it turns out that these are the only functions that satisfy this equation.

A few comments are worth making: First, unlike algebraic equations, (whose solutions are numbers), differential equations have solutions that are functions. We have seen above that depending on the constant \( k \), we get either functions with a positive or with a negative exponent (assuming that time \( t > 0 \)). This leads to the two distinct types of behaviour, exponential growth or exponential decay shown in Figures 9.1 and 9.2. In each of these figures we see a family of curves, each of which represents a function that satisfies one of the differential equations we have discussed.
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Figure 9.1: Functions of the form $y = Ce^{kt}$ for $k > 0$ represent exponentially growing solutions.

Figure 9.2: Functions of the form $y = Ce^{kt}$ for $k < 0$ represent exponentially decaying solutions.

9.2 The solution to a differential equation

Definition:

By a solution to a differential equation, we mean a function that satisfies that equation.

In the previous section we have seen a collection of solutions to each of the differential equations we discussed. For example, each of the curves shown in Figure 9.1 share the property that they satisfy the equation

$$\frac{dy}{dt} = ky.$$

We now ask: what distinguishes one from the other? More specifically, how could we specify one particular member of this family as the one of interest to us?

As we saw above, this is not done by the differential equation: we need some additional information. For example, if we gave some coordinates, say $(a, b)$ that the function of interest should go
through, this would select one out of the collection.

It is common practice (though not essential) to specify the starting value or initial value of the function i.e. its value at time $t = 0$.

**Definition:**

An initial value is the value at time $t = 0$ of the desired solution of a differential equation.

For example, suppose we are given

$$y(0) = y_0$$

where $y_0$ is some (known) fixed value. This will allow us to specify the unique value of the constant $C$ in the desired solution, as follows:

$$y(t) = Ce^{kt}$$

so

$$y(0) = Ce^{k\cdot0} = Ce^0 = C \cdot 1 = C$$

but, by the initial condition,

$$y(0) = y_0$$

So then,

$$C = y_0$$

and we have established that

$$y(t) = y_0e^{kt},$$

where $y_0$ is the initial value.

### 9.3 Where do differential equations come from?

Figure 9.3 shows how differential equations arise in scientific investigations. The process of going from initial vague observations about a system of interest (such as planetary motion) to a mathematical model, often involves a great deal of speculation, at first, about what is happening, what causes the motion or the changes that take place, and what assumptions might be fruitful in trying to analyze and understand the system.

Once the cloud of doubt and vague ideas settles somewhat, and once the right simplifying assumptions are made, we often find that the mathematical model leads to a differential equation. In most scientific applications, it may then be a huge struggle to figure out which functions would be the appropriate class of solutions to that differential equation, but if we can find those functions, we are in position to make quantitative predictions about the system of interest.

In our case, we have stumbled on a simple differential equation by noticing a property of functions that we were already familiar with. This is a lucky accident, and we will exploit it in an application shortly.

In many cases, the process of modelling hardly stops when we have found the link between the differential equation and solutions. Usually, we would then compare the predictions to observations that may help us to refine the model, reject incorrect or inaccurate assumptions, or determine to what extent the model has limitations.

A simple example of population growth modelling is given as motivation for some of the ideas seen in this discussion.
9.4 Population growth

In this section we will examine the way that a simple differential equation arises when we study the phenomenon of population growth.

We will let \( N(t) \) be the number of individuals in a population at time \( t \).

The population will change with time. Indeed the rate of change of \( N \) will be due to births (that increase \( N \)) and deaths (that decrease it).

\[
\text{Rate of change of } N = \text{Rate births} - \text{Rate deaths}
\]

We will assume that all individuals are identical in the population, and that the average *per capita birth rate*, \( r \), and average *per capita mortality rate*, \( m \) are some fixed positive constants. That is

\[
r = \text{per capita birth rate} = \frac{\text{number births per year}}{\text{population size}},
\]

\[
m = \text{per capita mortality rate} = \frac{\text{number deaths per year}}{\text{population size}}.
\]
We will refer to such constants as *parameters*. In general, for a given population, these would have certain numerical values that one could obtain by experiment, by observation, or by simple assumptions. In the next section, we will show how a set of assumptions would lead to such values.

Then the total number of births into the population in year $t$ is $rN$, and the total number of deaths out of the population in year $t$ is $mN$. The rate of change of the population as a whole is given by the derivative $dN/dt$. Thus we have arrived at:

$$\frac{dN}{dt} = rN - mN.$$ 

This is a differential equation: it links the derivative of $N(t)$ to the function $N(t)$. By solving the equation (i.e. identifying its solution), we will be able to make a projection about how fast the world population is growing.

We can first simplify the above by noting that

$$\frac{dN}{dt} = rN - mN = (r - m)N = kN.$$ 

where

$$k = (r - m).$$

This means that we have shown that the population satisfies a differential equation of the form

$$\frac{dN}{dt} = kN,$$

provided $k$ is the so-called “net growth rate”, i.e birth rate minus mortality rate. This leads us to the following conclusions:

- The function that describes population over time is (by previous results) simply

$$N(t) = N_0e^{kt}.$$ 

(The result is identical to what we saw previously, but with $N$ rather than $y$ as the time-dependent function.)

- We are no longer interested in negative values of $N$ since it now represents a quantity that has to be positive to have biological relevance, i.e. population size.

- The population will grow provided $k > 0$ which happens when $r - m > 0$ i.e. when the per capita birth rate, $r$ exceeds the per capita mortality rate $m$.

- If $k < 0$, or equivalently, $r < m$ then more people die on average than are born, so that the population will shrink and (eventually) go extinct.

### 9.5 Human population growth: a simple model

We have seen how a statement about changes that take place over time can lead to the formulation of a differential equation. In this section, we will estimate the values of the parameters for the birth rate, $r$ and the mortality rate, $m$.

To do so, we must make some simplifying assumptions:
Assumptions:

- The age distribution of the population is “flat”, i.e. there are as many 10 year-olds as 70 year-olds. (This is quite inaccurate, but will be a good place to start, as it will be easy to estimate some of the quantities we need.) Figure 9.4 shows such a distribution.

![Figure 9.4: We assume a “flat” age distribution to make it easy to determine the fraction of people who give birth or die.](image)

- The sex ratio is roughly 50%. This means that half of the population is female and half male.

- Women are fertile and can have babies only during part of their lives: We will assume that the fertile years are between age 15 and age 55, as shown in Figure 9.5.

![Figure 9.5: Only fertile women (between the ages of 15 and 55 years old) give birth. This sketch shows that half of all women are between these ages.](image)

- A lifetime lasts 80 years. This means that for half of that time a given woman can contribute to the birth rate, or that (55-15)/80=50% of women alive at any time are able to give birth.

- During a woman’s fertile years, we’ll assume that on average, she has one baby every 10 years. (This is also a suspect assumption, since in the Western world, a woman has on average 2-2.3 children over her lifetime, while in the Developing nations, the number of children per woman is much higher.)
Based on the above assumptions, we can estimate the parameter $r$ as follows:

$$r = \frac{\text{number women}}{\text{population}} \cdot \frac{\text{years fertile}}{\text{years of life}} \cdot \frac{\text{number babies per woman}}{\text{number of years}}$$

Thus we compute that

$$r = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{10} = 0.025 \text{ babies per person per year.}$$

Thus, we have arrived at an approximate value for human per capita birth rate.

We can now estimate the mortality.

- We also assume that deaths occur only from old age (i.e. we ignore disease, war, famine, and child mortality.)
- We assume that everyone lives precisely to age 80, and then dies instantly. (Not an assumption our grandparents would happily live with!)

\[ \text{number of people} \]

\[ \text{mortality occurs here} \]

\[ 0 \quad 80 \]

\[ \text{age} \]

Figure 9.6: We assume that the people in the age bracket 79-80 years old all die each year, and that those are the only deaths.

But, with the flat age distribution shown in Figure 9.4, there would be a fraction of 1/80 of the population who are precisely removed by mortality every year (i.e. only those of age 80.) In this case, we can estimate that the per capita mortality is:

$$m = \frac{1}{80} = 0.0125$$

Putting our results together, we have the net growth rate $k = r - m = 0.025 - 0.0125 = 0.0125$ per person per year. In the context of such growth problems, we will often refer to the constant $k$ as the *rate constant*, or the *growth rate* of the population.

We have found that our population satisfies the equation

$$\frac{dN}{dt} = 0.0125N$$

so that

$$N(t) = N_0e^{0.0125t}$$

where $N_0$ is the starting population size. Figure 9.7 illustrates how this function behaves, using a starting value of $N(0) = N_0 = 6$ billion.
9.6 Growth and doubling

We ask how long it would take for a population to double given that it is growing exponentially, with growth rate $k$, as described above. That is, we ask at what time $t$ it would be true that $n$ reaches twice its starting value, i.e. $N(t) = 2N_0$. We determine this time as follows:

$$N(t) = 2N_0$$

but

$$N(t) = N_0 e^{kt}$$

so the population has doubled when $t$ satisfies

$$2N_0 = N_0 e^{kt},$$

or simply

$$2 = e^{kt}$$

Taking the natural log of both sides leads to

$$\ln(2) = \ln(e^{kt}) = kt$$

Thus, the doubling time, which we’ll call $\tau$ is:

$$\tau = \frac{\ln(2)}{k}.$$

For example, for the growth rate $k = 0.0125$ found above, it will take

$$\tau = \frac{\ln(2)}{0.0125} = 55.45 \text{ years}$$
for the population to double. In general, an equation of the form

\[
\frac{dy}{dt} = ky
\]

that represents an exponential growth will have a doubling time of

\[
\tau = \frac{\ln(2)}{k}.
\]

This is shown in Figure 9.8.

![Graph showing exponential growth](image)

Figure 9.8:

The interesting thing that we discovered is that the population doubles *every 55 years*! So that, for example, after 110 years there have been two doublings, or a quadrupling of the population.

Exactly what does the model predict for the next 100 years? Suppose that currently \( N(0) = 6 \) billion. Then in billions,

\[
N(t) = 6e^{0.0125t}
\]

so that when \( t = 100 \) we would have

\[
N(100) = 6e^{0.0125\cdot100} = 6e^{1.25} = 6 \cdot 3.49 = 20.94
\]

Thus, with population around the 6 billion now, we should see about 21 billion people on Earth in 100 years.

### 9.7 A ten year doubling time

Suppose we are told that some animal population doubles every 10 years. What growth rate would lead to such a trend? Rearranging

\[
t_2 = \frac{\ln(2)}{k}.
\]
we obtain
\[ k = \frac{\ln(2)}{t_2} = \frac{0.6931}{10} \approx 0.07 \]
We may say that a growth rate of 7% leads to doubling roughly every 10 years.

9.8 A critique

Before leaving our population model, we should remember that our projections hold only so long as some rather restrictive assumptions are made. We have made many simplifications, and ignored many features that would seriously affect these results.

These include variations in the birth and mortality rates that stem from competition for the Earth’s resources, epidemics that take hold when crowding occurs, uneven distributions of resources or space, and other factors. We have also assumed that the age distribution is uniform (flat), but that is clearly wrong: the population grows only by adding new infants, and this would skew the distribution even if it starts out uniform. All these factors would lead us to be skeptical, and to eventually think about more advanced ways of describing the population growth.

9.9 Exponential decay and radioactivity

A radioactive material consists of atoms that undergo a spontaneous change. Every so often, an atom will emit a particle, and change to another form. We call this a process of radioactive decay.

For any one atom, it is impossible to predict when this event would occur. However, if we have very many atoms, on average some fraction, \( k \), will undergo this decay during any given unit time. (This fraction will depend on the material.) This means that \( ky \) of the amount will be lost per unit time.

We will define \( y(t) \) to be the amount of radioactivity remaining at time \( t \). This quantity can be measured with Geiger counters, and will depend on time. In the decay process, radioactivity will be continually lost. Thus

\[
\text{[rate of change of } y\text{]} = -\text{[amount lost per unit time]}
\]
or

\[
\frac{dy}{dt} = -ky
\]
We see again, a (by now) familiar differential equation.

Suppose that initially, there was an amount \( y_0 \). Then the initial condition that comes with this differential equation is

\[ y(0) = y_0. \]
From familiarity with the differential equation, we know that the function that satisfies it will be

\[ y(t) = Ce^{-kt} \]
and using the initial condition will specify that

\[ y(t) = y_0e^{-kt}. \]
For \( k > 0 \) a constant, this is a decreasing function of time that we refer to as exponential decay.
9.10 The half life

Given a process of exponential decay, we can ask how long it would take for half of the original amount to remain. That is, we look for \( t \) such that

\[
y(t) = \frac{y_0}{2}.
\]

We will refer to the value of \( t \) that satisfies this as the *half life*.

We compute:

\[
\frac{y_0}{2} = y_0 e^{-kt}
\]

\[
\frac{1}{2} = e^{-kt}
\]

Now taking reciprocals:

\[
2 = \frac{1}{e^{-kt}} = e^{kt}
\]

Thus, we find the same result as in our calculation for doubling times, namely,

\[
\ln(2) = \ln(e^{kt}) = kt
\]

so that the half life is

\[
\tau = \frac{\ln(2)}{k}
\]

This is shown in Figure 9.9.

![Figure 9.9](image)

9.11 Chernobyl: April 1986

In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. Of particular note were the two radioactive elements iodine-131 (I\(^{131}\)) whose half-life is 8 days and cesium-137 (Cs\(^{137}\)) whose half life is 30 years.
With our model for radioactive decay, we can predict how much of this material would remain over time. We first determine the decay constants for each of these two elements, by noting that

\[ k = \frac{\ln(2)}{\tau} \]

and recalling that \( \ln(2) \approx 0.693 \).

Then for \( ^{131}\text{I} \) we have

\[ k = \frac{\ln(2)}{\tau} = \frac{\ln(2)}{8} = 0.0866 \text{ per day}. \]

This means that if \( t \) is measured in days, then the amount of \( ^{131}\text{I} \) left at time \( t \) would be

\[ y_I(t) = y_0 e^{-0.0866t} \]

For \( ^{137}\text{Cs} \)

\[ k = \frac{\ln(2)}{30} = 0.023 \text{ per year}. \]

so that for \( t \) in years,

\[ y_C(t) = y_0 e^{-0.023t}. \]

If we ask how long it would take for \( ^{131}\text{I} \) to decay to 0.1 % of its initial level, just after the explosion at Chernobyl, we would calculate the time \( t \) such that \( y_I = 0.001y_0 \):

\[
0.001y_0 = y_0 e^{-0.0866t} \\
0.001 = e^{-0.0866t} \\
\ln(0.001) = -0.0866t \\
t = \frac{\ln(0.001)}{-0.0866} = \frac{-6.9}{-0.0866} = 79.7 \text{ days}
\]

Thus it would take about 80 days for the level of Iodine-131 to decay to 0.1 % of its initial level.

### 9.12 How do we determine the solution to a differential equation?

We have seen a number of examples of simple differential equations in this chapter, and our main purpose was to show how these arise in the context of a physical or biological process of growth or decay. Most of these examples led to the differential equation

\[ \frac{dy}{dt} = ky \]

and therefore, by our observations, to the exponential function

\[ y = f(t) = Ce^{kt}. \]

However, as we will see, there can be many distinct types of differential equations, and it may not always be clear which function is a solution. Finding the correct solution can be quite challenging, even to professional mathematicians. We mention two ideas that are sometimes helpful.
9.12.1 Verifying if a function is a solution to a differential equation

In some cases, we encounter a new differential equation, and we are given a function that is believed to satisfy that equation. We can always check and verify that this claim is correct (or find it incorrect) by simple differentiation.

Example: Newton’s Law of Cooling

Newton’s Law of Cooling states that the rate of change of the temperature of an object $T$, is proportional to the difference between the ambient (environment) temperature, $E$, and the temperature of the object, $T$. If the temperature of the environment is constant, and the object starts out at temperature $T_0$ initially, then the differential equation and initial condition describing this process is

$$\frac{dT}{dt} = k(E - T), \quad T(0) = T_0.$$ 

The parameter $k$ is a constant that represents the properties of the material. (Some objects conduct heat better than others, and thus cool off or heat up more quickly. The reader should be able to figure out that these types of objects have higher values of $k$, as this implies larger rates of change per unit time.) We study properties of this equation later, but here we show how to check which of two possible functions are its solutions.

Checking a solution: In Lab 4 of this calculus course, the students were told that the function below describes the temperature of a cooling object.

$$T(t) = E + (T_0 - E)e^{-kt}.$$ 

We now show that this function is a solution to the differential equation and initial value for Newton’s Law of cooling\(^1\).

Note, first, that by plugging in the initial time, $t = 0$, we have

$$T(0) = E + (T_0 - E)e^{-k\cdot0} = E + (T_0 - E) \cdot 1 = E + (T_0 - E) = T_0.$$ 

Thus the initial condition is satisfied.

Second, note that the derivative of this function is

$$\frac{dT}{dt} = \frac{d}{dt} (E + (T_0 - E)e^{-kt}) = -k(T_0 - E)e^{-kt}.$$

(This follows from the fact that $E$ is a constant, $(T_0 - E)$ is constant, and from the chain rule applied to the exponent $-kt$.) The term on the right hand side of the differential equation leads to

$$k(E - T) = k(E - [E + (T_0 - E)e^{-kt}]) = -k(T_0 - E)e^{-kt}.$$ 

We now observe agreement between the terms obtained from each of the right and left hand sides of the differential equation, applied to the above function. We conclude that the differential equation is satisfied, so that indeed this candidate function is a solution, as claimed.

As shown in this example, if we are told that a function is a solution to a differential equation, we can check the assertion and verify that it is correct or incorrect. A much more difficult task is to

\(^1\)In Lab 4, this formula was presented with no discussion of its related differential equation. Here we see that relationship for the first time.
find the solution of a new differential equation from first principles. In some cases, the technique of integration, learned in second semester calculus, can be used. In other cases, some transformation that changes the problem to a more familiar one is helpful. (An example of this type is presented in Chapter 13). In many cases, particularly those of so-called non-linear differential equations, it requires great expertise and familiarity with advanced mathematical methods to find the solution to such problems in an analytic form, i.e. as an explicit formula. In such cases, approximation and numerical methods are helpful.

9.13 How do we solve a differential equation approximately?

In cases where it is difficult or impossible to find the desired solution with guesses, integration methods, or from previous experience, we can use approximation methods and numerical computations to do the job. Most of these methods rely on the fact that derivatives can be approximated by finite differences. For example, suppose we are given a differential equation of the form

$$\frac{dy}{dt} = f(y)$$

with initial value $y(0) = y_0$, can be approximated by selecting a set of time points $t_1, t_2, \ldots$, which are spaced apart by time steps of size $\Delta t$, and replacing the differential equation by the approximate finite difference equation

$$\frac{y_1 - y_0}{\Delta t} = f(y_0).$$

This relies on the approximation

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t},$$

which is a relatively good approximation for small step size $\Delta t$. Then by rearranging this approximation, we find that

$$y_1 = y_0 + f(y_0)\Delta t.$$

Knowing the quantities on the right allows us to compute the value of $y_1$, i.e. the value of the approximate “solution” at the time point $t_1$. We can then continue to generate the value at the next time point in the same way, by approximating the derivative again as a secant slope. This leads to

$$y_2 = y_1 + f(y_1)\Delta t.$$

The approximation so generated, leading to values $y_1, y_2, \ldots$ is called Euler’s method. We explore an application of this method to Newton’s law of cooling in chapter 13. In Lab 5, the reader is invited to try out this method on the differential equation for Newton’s Law of Cooling that was discussed in this chapter.

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2We will devote an entire chapter to discussing such approximation methods (Chapter 12). Here we preview one technique, Euler’s method, as it is encountered in Lab 5 before Chapter 12 is finished.