Simpson’s 1/3 Rule of Integration
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After reading this chapter, you should be able to

1. derive the formula for Simpson’s 1/3 rule of integration,
2. use Simpson’s 1/3 rule it to solve integrals,
3. develop the formula for multiple-segment Simpson’s 1/3 rule of integration,
4. use multiple-segment Simpson’s 1/3 rule of integration to solve integrals, and
5. derive the true error formula for multiple-segment Simpson’s 1/3 rule.

What is integration?
Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson’s 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson’s 1/3 rule of approximating integrals of the form

\[ I = \int_{a}^{b} f(x) \, dx \]

where

- \( f(x) \) is called the integrand,
- \( a = \) lower limit of integration
- \( b = \) upper limit of integration
Simpson’s 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson’s 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

**Figure 1** Integration of a function

Method 1:

Hence

\[ I = \int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} f_2(x) \, dx \]

where \( f_2(x) \) is a second order polynomial given by

\[ f_2(x) = a_0 + a_1 x + a_2 x^2. \]

Choose

\[ (a, f(a)), \left( \frac{a + b}{2}, f\left( \frac{a + b}{2} \right) \right), \text{ and } (b, f(b)) \]
as the three points of the function to evaluate $a_0$, $a_1$ and $a_2$.

\[ f(a) = f_3(a) = a_0 + a_1a + a_2a^2 \]
\[ f\left(\frac{a + b}{2}\right) = f_2\left(\frac{a + b}{2}\right) = a_0 + a_1\left(\frac{a + b}{2}\right) + a_2\left(\frac{a + b}{2}\right)^2 \]
\[ f(b) = f_3(b) = a_0 + a_1b + a_2b^2 \]

Solving the above three equations for unknowns, $a_0$, $a_1$ and $a_2$ give

\[ a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a + b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2} \]
\[ a_1 = -\frac{af(a) - 4af\left(\frac{a + b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a + b}{2}\right) + bf(b)}{a^2 - 2ab + b^2} \]
\[ a_2 = \frac{2\left(f(a) - 2f\left(\frac{a + b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2} \]

Then

\[ I \approx \int_a^b f_2(x)dx \]
\[ = \int_a^b (a_0 + a_1x + a_2x^2)dx \]
\[ = \left[a_0x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]^b_a \]
\[ = a_0(b - a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3} \]

Substituting values of $a_0$, $a_1$ and $a_2$ give
\[ \int_{a}^{b} f_{2}(x) \, dx = \frac{b-a}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \]

Since for Simpson 1/3 rule, the interval \( [a, b] \) is broken into 2 segments, the segment width

\[ h = \frac{b-a}{2} \]

Hence the Simpson’s 1/3 rule is given by

\[ \int_{a}^{b} f(x) \, dx \approx \frac{h}{3} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] \]

Since the above form has 1/3 in its formula, it is called Simpson’s 1/3 rule.

**Method 2:**
Simpson’s 1/3 rule can also be derived by approximating \( f(x) \) by a second order polynomial using Newton's divided difference polynomial as

\[ f_{2}(x) = b_{0} + b_{1}(x-a) + b_{2}(x-a)\left(x-\frac{a+b}{2}\right) \]

where

\[ b_{0} = f(a) \]

\[ b_{1} = \frac{f \left( \frac{a+b}{2} \right) - f(a)}{\frac{a+b}{2} - a} \]

\[ b_{2} = \frac{f(b) - f \left( \frac{a+b}{2} \right) - f \left( \frac{a+b}{2} \right) - f(a)}{\frac{b-a}{2} - \frac{a+b}{2} - a} \]
Integrating Newton’s divided difference polynomial gives us

\[
\int_a^b f(x) \, dx \approx \int_a^b f_2(x) \, dx
\]

\[
= \int_a^b \left[ b_0 + b_1(x-a) + b_2(x-a)^2 \right] \, dx
\]

\[
= \left[ b_0 x + b_1 \left( \frac{x^2}{2} - ax \right) + b_2 \left( \frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2} \right) \right]_a^b
\]

\[
= b_0 (b-a) + \left( b_1 \frac{b^2-a^2}{2} - a(b-a) \right) + b_2 \left( \frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2} \right)
\]

Substituting values of \( b_0, b_1, \) and \( b_2 \) into this equation yields the same result as before

\[
\int_a^b f(x) \, dx \approx \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right]
\]

\[
= \frac{h}{3} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right]
\]

Method 3:
One could even use the Lagrange polynomial to derive Simpson’s formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes
\[ f_2(x) = \frac{(x-a+b)}{2} \frac{(x-b)}{(a-b)} f(a) + \frac{(x-a)(x-b)}{(a+b-2)(a+b-b)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)(x-a+b)}{(b-a)(b-a+b)} f(b) \]

Integrating this function gets

\[
\int_{a}^{b} f_2(x) \, dx = \left[ \frac{x^3}{3} - \frac{(a+3b)x^2}{4} + \frac{b(a+b)x}{2} \frac{(a-a+b)}{2}(a-b) \right]_{a}^{b}
\]

\[
= \frac{b^3-a^3}{3} - \frac{(a+3b)(b^2-a^2)}{4} + \frac{b(a+b)(b-a)}{(a-a+b)} f(a)
\]

\[
+ \frac{b^3-a^3}{3} - \frac{(a+b)(b^2-a^2)}{2} + ab(b-a) f\left(\frac{a+b}{2}\right)
\]

\[
= \frac{b^3-a^3}{3} - \frac{(a+b)(b^2-a^2)}{2} + ab(b-a) f\left(\frac{a+b}{2}\right)
\]

Believe it or not, simplifying and factoring this large expression yields you the same result as before

\[
\int_{a}^{b} f(x) \, dx \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]
\[
= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].
\]

Method 4:
Simpson’s 1/3 rule can also be derived by the method of coefficients. Assume
\[
\int_a^b f(x)\,dx \approx c_1f(a) + c_2f\left(\frac{a+b}{2}\right) + c_3f(b)
\]
Let the right-hand side be an exact expression for the integrals \(\int_a^b 1\,dx\), \(\int_a^b x\,dx\), and \(\int_a^b x^2\,dx\).
This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now
\[
\int_a^b 1\,dx = b - a = c_1 + c_2 + c_3
\]
\[
\int_a^b x\,dx = \frac{b^2 - a^2}{2} = c_1a + c_2\frac{a + b}{2} + c_3b
\]
\[
\int_a^b x^2\,dx = \frac{b^3 - a^3}{3} = c_1a^2 + c_2\left(\frac{a + b}{2}\right)^2 + c_3b^2
\]
Solving the above three equations for \(c_0\), \(c_1\), and \(c_2\) give
\[
c_1 = \frac{b - a}{6}
\]
\[
c_2 = \frac{2(b - a)}{3}
\]
\[
c_3 = \frac{b - a}{6}
\]
This gives
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\[
\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)
\]

\[
= \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

\[
= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

The integral from the first method

\[
\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1x + a_2x^2)dx
\]

can be viewed as the area under the second order polynomial, while the equation from Method 4

\[
\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)
\]

can be viewed as the sum of the areas of three rectangles.

**Example 1**

The distance covered by a rocket in meters from \( t = 8 \) s to \( t = 30 \) s is given by

\[
x = \int_8^{30} \left( 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right) dt
\]

a) Use Simpson’s 1/3 rule to find the approximate value of \( x \).

b) Find the true error, \( E_1 \).

c) Find the absolute relative true error, \( |\varepsilon| \).

**Solution**

a) \( x \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \)
\[ a = 8 \]
\[ b = 30 \]
\[ \frac{a + b}{2} = 19 \]

\[ f(t) = 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \]

\[ f(8) = 2000 \ln \left[ \frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 \text{ m/s} \]

\[ f(30) = 2000 \ln \left[ \frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s} \]

\[ f(19) = 2000 \ln \left( \frac{140000}{140000 - 2100(19)} \right) - 9.8(19) = 484.75 \text{ m/s} \]

\[ x \approx \frac{b - a}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] \]

\[ = \left( \frac{30 - 8}{6} \right) \left[ f(8) + 4f(19) + f(30) \right] \]

\[ = \frac{22}{6} \left[ 177.27 + 4 \times 484.75 + 901.67 \right] \]

\[ = 11065.72 \text{ m} \]

b) The exact value of the above integral is
\[
x = \int_{a}^{b} \left[ 2000 \ln \left( \frac{140000}{140000 - 2100r} \right) - 9.8t \right] dt
\]

= 11061.34 m

So the true error is

\[
E_t = True Value - Approximate Value
\]

= 11061.34 - 11065.72

= -4.38 m

c) Absolute Relative true error,

\[
|\varepsilon_t| = \left| \frac{True Error}{True Value} \right| \times 100
\]

= \left| \frac{-4.38}{11061.34} \right| \times 100

= 0.0396%

**Multiple-segment Simpson’s 1/3 Rule**

Just like in multiple-segment trapezoidal rule, one can subdivide the interval \([a, b]\) into \(n\) segments and apply Simpson’s 1/3 rule repeatedly over every two segments. Note that \(n\) needs to be even. Divide interval \([a, b]\) into \(n\) equal segments, so that the segment width is given by

\[
h = \frac{b - a}{n}.
\]

Now

\[
\int_{a}^{b} f(x) dx = \sum_{i=0}^{n-1} f(x_i) dx
\]

where

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\[ x_0 = a \]
\[ x_n = b \]

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{x_1} f(x) \, dx + \int_{x_1}^{x_2} f(x) \, dx + \ldots + \int_{x_{n-4}}^{x_{n-2}} f(x) \, dx + \int_{x_{n-2}}^{x_n} f(x) \, dx \]

Apply Simpson’s 1/3rd Rule over each interval,

\[ \int_{a}^{b} f(x) \, dx \cong (x_2 - x_0) \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + (x_4 - x_2) \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \ldots \]

\[ + (x_{n-2} - x_{n-4}) \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + (x_n - x_{n-2}) \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right] \]

Since

\[ x_i - x_{i-2} = 2h \]
\[ i = 2, 4, \ldots, n \]

then

\[ \int_{a}^{b} f(x) \, dx \cong 2h \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + 2h \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \ldots \]

\[ + 2h \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + 2h \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right] \]

\[ = \frac{h}{3} \left[ f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\} + f(x_n) \right] \]
\[ x = \frac{b - a}{3n} \left[ \int_{a}^{b} f(x) \, dx \approx f(t_0) + \frac{4}{3} \sum_{i=1 \text{ odd}}^{n-1} f(t_i) + 2 \sum_{i=2 \text{ even}}^{n-2} f(t_i) + f(t_n) \right] \]

**Example 2**

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from \( t = 8 \) s to \( t = 30 \) s as given by

\[ x = \int_{t}^{30} \left( 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right) \, dt \]

a) Use four segment Simpson's 1/3rd Rule to find the probability.

b) Find the true error, \( E_t \), for part (a).

c) Find the absolute relative true error, \( |\varepsilon| \), for part (a).

**Solution:**

a) Using \( n \) segment Simpson's 1/3 rule,

\[ x \approx \frac{b - a}{3n} \left[ f(t_0) + \frac{4}{3} \sum_{i=1 \text{ odd}}^{n-1} f(t_i) + 2 \sum_{i=2 \text{ even}}^{n-2} f(t_i) + f(t_n) \right] \]

\[ n = 4 \]

\[ a = 8 \]

\[ b = 30 \]

\[ h = \frac{b - a}{n} \]

\[ = \frac{30 - 8}{4} \]
\[ f(t) = 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \]

So

\[ f(t_0) = f(8) \]

\[ f(8) = 2000 \ln \left( \frac{140000}{140000 - 2100(8)} \right) - 9.8(8) = 177.27 \text{ m/s} \]

\[ f(t_1) = f(8 + 5.5) = f(13.5) \]

\[ f(13.5) = 2000 \ln \left( \frac{140000}{140000 - 2100(13.5)} \right) - 9.8(13.5) = 320.25 \text{ m/s} \]

\[ f(t_2) = f(13.5 + 5.5) = f(19) \]

\[ f(19) = 2000 \ln \left( \frac{140000}{140000 - 2100(19)} \right) - 9.8(19) = 484.75 \text{ m/s} \]

\[ f(t_3) = f(19 + 5.5) = f(24.5) \]

\[ f(24.5) = 2000 \ln \left( \frac{140000}{140000 - 2100(24.5)} \right) - 9.8(24.5) = 676.05 \text{ m/s} \]
\[ f(t_4) = f(t_n) = f(30) \]

\[ f(30) = 2000 \ln \left[ \frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s} \]

\[ x = \frac{b - a}{3n} \left[ f(t_0) + 4 \sum_{i=1 \atop i=\text{odd}}^{n-1} f(t_i) + 2 \sum_{i=2 \atop i=\text{even}}^{n-2} f(t_i) + f(t_n) \right] \]

\[ = \frac{30 - 8}{3(4)} \left[ f(8) + 4 \sum_{i=1 \atop i=\text{odd}}^{3} f(t_i) + 2 \sum_{i=2 \atop i=\text{even}}^{2} f(t_i) + f(30) \right] \]

\[ = \frac{22}{12} \left[ f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right] \]

\[ = \frac{11}{6} \left[ f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30) \right] \]

\[ = \frac{11}{6} \left[ 177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67 \right] \]

\[ = 11061.64 \text{ m} \]

b) The exact value of the above integral is

\[ \int_{8}^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \]

\[ = 11061.34 \text{ m} \]

So the true error is

\[ E_t = \text{True Value} - \text{Approximate Value} \]

\[ E_t = 11061.34 - 11061.64 \]
\[ = -0.30 \text{ m} \]

c) Absolute Relative true error,

\[ |\varepsilon_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \]

\[ = \left| \frac{-0.3}{11061.34} \right| \times 100 \]

\[ = 0.0027\% \]

**Table 1** Values of Simpson's 1/3 rule for Example 2 with multiple-segments

| \( n \) | Approximate Value | \( E_i \) | \( |\varepsilon_t| \) |
|---|---|---|---|
| 2 | 11065.72 | -4.38 | 0.0396\% |
| 4 | 11061.64 | -0.30 | 0.0027\% |
| 6 | 11061.40 | -0.06 | 0.0005\% |
| 8 | 11061.35 | -0.02 | 0.0002\% |
| 10 | 11061.34 | -0.01 | 0.0001\% |

**Error in Multiple-segment Simpson's 1/3 rule**

The true error in a single application of Simpson's 1/3rd Rule is given\(^1\) by

\[ E_i = \frac{(b - a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b \]

In multiple-segment Simpson’s 1/3 rule, the error is the sum of the errors in each application of Simpson’s 1/3 rule. The error in the \( n \) segments Simpson's 1/3rd Rule is given by

\(^1\) The \( f^{(4)} \) in the true error expression stands for the fourth derivative of the function \( f(x) \).

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\[ E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \]
\[ = -\frac{h^5}{90} f^{(4)}(\zeta_1) \]

\[ E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \]
\[ = -\frac{h^5}{90} f^{(4)}(\zeta_2) \]

\[ \vdots \]

\[ E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i} \]
\[ = -\frac{h^5}{90} f^{(4)}(\zeta_i) \]

\[ \vdots \]

\[ E_{\frac{n}{2} - 1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\frac{\zeta_{n-2}}{2}\right), \quad x_{n-4} < \frac{\zeta_{n-2}}{2} < x_{n-2} \]
\[ = -\frac{h^5}{90} f^{(4)}\left(\frac{\zeta_{n-2}}{2}\right) \]

\[ E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\frac{\zeta_n}{2}\right), \quad x_{n-2} < \frac{\zeta_n}{2} < x_n \]

Hence, the total error in the multiple-segment Simpson’s 1/3 rule is

\[ = -\frac{h^5}{90} f^{(4)}\left(\frac{\zeta_n}{2}\right) \]

\[ E_t = \sum_{i=1}^{n} E_i \]
\[ E_i = -\frac{(b - a)^5}{90n^4} \bar{f}^{(4)} \]

where

\[ \bar{f}^{(4)} = \frac{1}{n} \sum_{i=1}^{n} f^{(4)}(\zeta_i) \]

The term \( \frac{1}{n} \sum_{i=1}^{n} f^{(4)}(\zeta_i) \) is an approximate average value of \( f^{(4)}(x) \), \( a < x < b \). Hence

\[ E_i = -\frac{(b - a)^5}{90n^4} \bar{f}^{(4)} \]