**Divisor function**

In mathematics, and specifically in number theory, a divisor function is an arithmetical function related to the divisors of an integer. When referred to as the divisor function, it counts the number of divisors of an integer. It appears in a number of remarkable identities, including relationships on the Riemann zeta function and the Eisenstein series of modular forms. Divisor functions were studied by Ramanujan, who gave a number of important congruences and identities.

A related function is the divisor summatory function, which, as the name implies, is a sum over the divisor function.

**Definition**

The sum of positive divisors function \( \sigma_x(n) \), for a real or complex number \( x \), is defined as the sum of the \( x \)th powers of the positive divisors of \( n \), or

\[
\sigma_x(n) = \sum_{d|n} d^x.
\]

The notations \( d(n) \), \( v(n) \) and \( \tau(n) \) (for the German Teiler = divisors) are also used to denote \( \sigma_0(n) \), or the number-of-divisors function \([1][2]\) (sequence A000005 in OEIS). When \( x \) is 1, the function is called the sigma function or sum-of-divisors function, \([3][4]\] and the subscript is often omitted, so \( \sigma(n) \) is equivalent to \( \sigma_1(n) \) ( A000203).

The aliquot sum \( s(n) \) of \( n \) is the sum of the proper divisors (that is, the divisors excluding \( n \) itself, \( \text{A001065} \)), and equals \( \sigma_1(n) - n \); the aliquot sequence of \( n \) is formed by repeatedly applying the aliquot sum function.

**Example**

For example, \( \sigma_0(12) \) is the number of the divisors of 12:

\[
\sigma_0(12) = 1^0 + 2^0 + 3^0 + 4^0 + 6^0 + 12^0 = 1 + 1 + 1 + 1 + 1 + 1 = 6,
\]

while \( \sigma_1(12) \) is the sum of all the divisors:

\[
\sigma_1(12) = 1^1 + 2^1 + 3^1 + 4^1 + 6^1 + 12^1 = 1 + 2 + 3 + 4 + 6 + 12 = 28,
\]

and the aliquot sum \( s(12) \) of proper divisors is:

\[
s(12) = 1^1 + 2^1 + 3^1 + 4^1 + 6^1 = 1 + 2 + 3 + 4 + 6 = 16.
\]
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The cases x=2, x=3 and so on are tabulated in A001157, A001158, A001159, A001160, A013954, A013955 ...
Properties

For a non-square integer every divisor $d$ of $n$ is paired with divisor $n/d$ of $n$ and $\sigma_0(n)$ is then even; for a square integer one divisor (namely $\sqrt{n}$) is not paired with a distinct divisor and $\sigma_0(n)$ is then odd.

For a prime number $p$,
\[
\begin{align*}
    d(p) &= 2 \\
    d(p^n) &= n + 1 \\
    \sigma(p) &= p + 1
\end{align*}
\]
because by definition, the factors of a prime number are 1 and itself. Also, where $p_n#$ denotes the primorial,
\[
d(p_n#) = 2^n
\]
since $n$ prime factors allow a sequence of binary selection ($P_i$ or 1) from $n$ terms for each proper divisor formed.

Clearly, $1 < d(n) < n$ and $\sigma(n) > n$ for all $n > 2$.

The divisor function is multiplicative, but not completely multiplicative. The consequence of this is that, if we write
\[
n = \prod_{i=1}^{r} p_i^{a_i}
\]
where $r = \omega(n)$ is the number of distinct prime factors of $n$, $p_i$ is the $i$th prime factor, and $a_i$ is the maximum power of $p_i$ by which $n$ is divisible, then we have
\[
\sigma_x(n) = \prod_{i=1}^{r} \frac{p_i^{(a_i+1)x}}{p_i^x - 1}
\]
which is equivalent to the useful formula:
\[
\sigma_x(n) = \prod_{i=1}^{r} \sum_{j=0}^{a_i} p_i^{jx} = \prod_{i=1}^{r} \left(1 + p_i^x + p_i^{2x} + \cdots + p_i^{a_ix}\right)
\]
It follows (by setting $x = 0$) that $d(n)$ is:
\[
d(n) = \prod_{i=1}^{r} (a_i + 1).
\]
For example, if $n$ is 24, there are two prime factors ($p_1$ is 2; $p_2$ is 3); noting that 24 is the product of $2^3 \times 3^1$, $a_1$ is 3 and $a_2$ is 1. Thus we can calculate $d(24)$ as so:
\[
d(24) = \prod_{i=1}^{2} (a_i + 1)
\]
\[
= (3 + 1)(1 + 1) = 4 \times 2 = 8.
\]
The eight divisors counted by this formula are 1, 2, 4, 8, 3, 6, 12, and 24.

We also note $s(n) = \sigma(n) - n$. Here $s(n)$ denotes the sum of the proper divisors of $n$, i.e. the divisors of $n$ excluding $n$ itself. This function is the one used to recognize perfect numbers which are the $n$ for which $s(n) = n$. If $s(n) > n$ then $n$ is an abundant number and if $s(n) < n$ then $n$ is a deficient number.

If $n$ is a power of 2, e.g. $n = 2^k$, then $\sigma(n) = 2 \times 2^k - 1 = 2n - 1$, and $s(n) = n - 1$, which makes $n$ almost-perfect.

As an example, for two distinct primes $p$ and $q$ with $p < q$, let
\[
n = pq.
\]
Then
\[
\sigma(n) = (p + 1)(q + 1) = n + 1 + (p + q),
\]
\[
\phi(n) = (p - 1)(q - 1) = n + 1 - (p + q),
\]
and

\[
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\]
\[
3
\]
\[ n + 1 = (\sigma(n) + \phi(n))/2, \]
\[ p + q = (\sigma(n) - \phi(n))/2, \]
where \( \phi(n) \) is Euler's totient function.

Then, the roots of:
\[(x-p)(x-q) = x^2 - (p+q)x + n = x^2 - [(\sigma(n) - \phi(n))/2]x + [(\sigma(n) + \phi(n))/2 - 1] = 0\]
allows us to express \( p \) and \( q \) in terms of \( \sigma(n) \) and \( \phi(n) \) only, without even knowing \( n \) or \( p+q \), as:
\[ p = (\sigma(n) - \phi(n))/4 - \sqrt{[(\sigma(n) - \phi(n))/4]^2 - [(\sigma(n) + \phi(n))/2 - 1]}, \]
\[ q = (\sigma(n) - \phi(n))/4 + \sqrt{[(\sigma(n) - \phi(n))/4]^2 - [(\sigma(n) + \phi(n))/2 - 1]}. \]
Also, knowing \( n \) and either \( \sigma(n) \) or \( \phi(n) \) (or knowing \( p+q \) and either \( \sigma(n) \) or \( \phi(n) \)) allows us to easily find \( p \) and \( q \).

In 1984, Roger Heath-Brown proved that
\[ d(n) = d(n+1) \]
will occur infinitely often.

### Series relations

Two Dirichlet series involving the divisor function are:
\[ \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \zeta(s) \zeta(s-a), \]
which for \( d(n) = \sigma_0(n) \) gives
\[ \sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s), \]
and
\[ \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}. \]

A Lambert series involving the divisor function is:
\[ \sum_{n=1}^{\infty} q^n \sigma_a(n) = \sum_{n=1}^{\infty} \frac{n^a q^n}{1 - q^n} \]
for arbitrary complex \( |q| \leq 1 \) and \( a \). This summation also appears as the Fourier series of the Eisenstein series and the invariants of the Weierstrass elliptic functions.

### Approximate growth rate

In little-o notation, the divisor function satisfies the inequality (see page 296 of Apostol’s book\(^5\))
\[ \text{for all } \epsilon > 0, \quad d(n) = o(n^\epsilon). \]
More precisely, Severin Wigert showed that
\[ \lim_{n \to \infty} \sup \frac{\log d(n)}{\log n / \log \log n} = \log 2. \]
On the other hand, since there are infinitely many prime numbers,
\[ \lim_{n \to \infty} \inf d(n) = 2. \]
In Big-O notation, Dirichlet showed that the average order of the divisor function satisfies the following inequality (see Theorem 3.3 of Apostol’s book\(^5\))
for all \( x \geq 1 \), \( \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}) \),

where \( \gamma \) is Euler's constant. Improving the bound \( O(\sqrt{x}) \) in this formula is known as Dirichlet's divisor problem.

The behaviour of the sigma function is irregular. The asymptotic growth rate of the sigma function can be expressed by:

\[
\limsup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma,
\]

where \( \limsup \) is the limit superior. This result is **Grönwall's theorem**, published in 1913 (Grönwall 1913). His proof uses Mertens' 3rd theorem, which says that

\[
\lim_{n \to \infty} \frac{1}{\log n} \prod_{p \leq n} \frac{p}{p - 1} = e^\gamma,
\]

where \( p \) denotes a prime.

In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality:

\[
\sigma(n) < e^n n \log \log n \quad \text{(Robin's inequality)}
\]

holds for all sufficiently large \( n \). In 1984 Guy Robin proved that the inequality is true for all \( n \geq 5,041 \) if and only if the Riemann hypothesis is true (Robin 1984). This is **Robin's theorem** and the inequality became known after him. The largest known value that violates the inequality is \( n=5,040 \). If the Riemann hypothesis is true, there are no greater exceptions. If the hypothesis is false, then Robin showed there are an infinite number of values of \( n \) that violate the inequality, and it is known that the smallest such \( n \geq 5,041 \) must be superabundant (Akbary & Friggstad 2009). It has been shown that the inequality holds for large odd and square-free integers, and that the Riemann hypothesis is equivalent to the inequality just for \( n \) divisible by the fifth power of a prime (Choie et al. 2007).

A related bound was given by Jeffrey Lagarias in 2002, who proved that the Riemann hypothesis is equivalent to the statement that

\[
\sigma(n) \leq H_n + \ln(H_n)e^{H_n}
\]

for every natural number \( n \), where \( H_n \) is the \( n \)th harmonic number, (Lagarias 2002).

Robin also proved, unconditionally, that the inequality

\[
\sigma(n) < e^n n \log \log n + \frac{0.6483}{\log \log n} n
\]

holds for all \( n \geq 3 \).

**Notes**

[1] Long (1972, p. 46)


**References**


• Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions (http://mathstat.carleton.ca/~williams/papers/pdf/249.pdf) PDF of a paper by Huard, Ou, Spearman, and Williams. Contains elementary (i.e. not relying on the theory of modular forms) proofs of divisor sum convolutions, formulas for the number of ways of representing a number as a sum of triangular numbers, and related results.
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