In statistics, the Kalman filter is a mathematical method named after Rudolf E. Kalman. Its purpose is to use measurements that are observed over time that contain noise (random variations) and other inaccuracies, and produce values that tend to be closer to the true values of the measurements and their associated calculated values. The Kalman filter has many applications in technology, and is an essential part of the development of space and military technology. Perhaps the most commonly used type of very simple Kalman filter is the phase-locked loop, which is now ubiquitous in FM radios and most electronic communications equipment. Extensions and generalizations to the method have also been developed.

The Kalman filter produces estimates of the true values of measurements and their associated calculated values by predicting a value, estimating the uncertainty of the predicted value, and computing a weighted average of the predicted value and the measured value. The most weight is given to the value with the least uncertainty. The estimates produced by the method tend to be closer to the true values than the original measurements because the weighted average has a better estimated uncertainty than either of the values that went into the weighted average.

From a theoretical standpoint, the Kalman filter is an algorithm for efficiently doing exact inference in a linear dynamical system, which is a Bayesian model similar to a hidden Markov model but where the state space of the latent variables is continuous and where all latent and observed variables have a Gaussian distribution (often a multivariate Gaussian distribution).

### Naming and historical development

The filter is named after Rudolf E. Kalman, though Thorvald Nicolai Thiele\[^{1}\]\[^{2}\] and Peter Swerling developed a similar algorithm earlier. Richard S. Bucy of the University of Southern California contributed to the theory, leading to it often being called the Kalman-Bucy filter. Stanley F. Schmidt is generally credited with developing the first implementation of a Kalman filter. It was during a visit by Kalman to the NASA Ames Research Center that he saw the applicability of his ideas to the problem of trajectory estimation for the Apollo program, leading to its incorporation in the Apollo navigation computer. This Kalman filter was first described and partially developed in technical papers by Swerling (1958), Kalman (1960) and Kalman and Bucy (1961).

Kalman filters have been vital in the implementation of the navigation systems of U.S. Navy nuclear ballistic missile submarines, and in the guidance and navigation systems of cruise missiles such as the U.S. Navy's Tomahawk missile and the U.S. Air Force's Air Launched Cruise Missile. It is also used in the guidance and navigation systems of the NASA Space Shuttle and the attitude control and navigation systems of the International Space Station.

This digital filter is sometimes called the \textit{Stratonovich–Kalman–Bucy filter} because it is a special case of a more general, non-linear filter developed somewhat earlier by the Soviet mathematician Ruslan L. Stratonovich.\[^{3}\]\[^{4}\] In fact, some of the special case linear filter's equations appeared in these papers by Stratonovich that were published before summer 1960, when Kalman met with Stratonovich during a conference in Moscow.
Overview of the calculation

The Kalman filter uses a system's dynamics model (i.e., physical laws of motion), known control inputs to that system, and measurements (such as from sensors) to form an estimate of the system's varying quantities (its state) that is better than the estimate obtained by using any one measurement alone. As such, it is a common sensor fusion algorithm.

All measurements and calculations based on models are estimates to some degree. Noisy sensor data, approximations in the equations that describe how a system changes, and external factors that are not accounted for introduce some uncertainty about the inferred values for a system's state. The Kalman filter averages a prediction of a system's state with a new measurement using a weighted average. The purpose of the weights is that values with better (i.e., smaller) estimated uncertainty are "trusted" more. The weights are calculated from the covariance, a measure of the estimated uncertainty of the prediction of the system's state. The result of the weighted average is a new state estimate that lies in between the predicted and measured state, and has a better estimated uncertainty than either alone. This process is repeated every time step, with the new estimate and its covariance informing the prediction used in the following iteration. This means that the Kalman filter works recursively and requires only the last "best guess" - not the entire history - of a system's state to calculate a new state.

When performing the actual calculations for the filter (as discussed below), the state estimate and covariances are coded into matrices to handle the multiple dimensions involved in a single set of calculations. This allows for representation of linear relationships between different state variables (such as position, velocity, and acceleration) in any of the transition models or covariances.

Example application

The Kalman filter is used in sensor fusion and data fusion. Typically real time systems produce multiple sequential measurements rather than making a single measurement to obtain the state of the system. These multiple measurements are then combined mathematically to generate the system's state at that time instant.

As an example application, consider the problem of determining the precise location of a truck. The truck can be equipped with a GPS unit that provides an estimate of the position within a few meters. The GPS estimate is likely to be noisy; readings 'jump around' rapidly, though always remaining within a few meters of the real position. The truck's position can also be estimated by integrating its speed and direction over time, determined by keeping track of the amount the accelerator is depressed and how much the steering wheel is turned. This is a technique known as dead reckoning. Typically, dead reckoning will provide a very smooth estimate of the truck's position, but it will drift over time as small errors accumulate. Additionally, the truck is expected to follow the laws of physics, so its position should be expected to change proportionally to its velocity.

In this example, the Kalman filter can be thought of as operating in two distinct phases: predict and update. In the prediction phase, the truck's old position will be modified according to the physical laws of motion (the dynamic or "state transition" model) plus any changes produced by the accelerator pedal and steering wheel. Not only will a new position estimate be calculated, but a new covariance will be calculated as well. Perhaps the covariance is proportional to the speed of the truck because we are more uncertain about the accuracy of the dead reckoning estimate at high speeds but very certain about the position when moving slowly. Next, in the update phase, a measurement of the truck's position is taken from the GPS unit. Along with this measurement comes some amount of uncertainty, and its covariance relative to that of the prediction from the previous phase determines how much the new measurement will affect the updated prediction. Ideally, if the dead reckoning estimates tend to drift away from the real position, the GPS measurement should pull the position estimate back towards the real position but not disturb it to the point of becoming rapidly changing and noisy.
Kalman filter in computer vision

Data fusion using a Kalman filter can assist computers to track objects in videos with low latency. The tracking of objects is a dynamic problem, using data from sensor and camera images that always suffer from noise. This can sometimes be reduced by using higher quality cameras and sensors but can never be eliminated, so it is often desirable to use a noise reduction method.

The iterative predictor-corrector nature of the Kalman filter can be helpful, because at each time instance only one constraint on the state variable need be considered. This process is repeated, considering a different constraint at every time instance. All the measured data are accumulated over time and help in predicting the state.

Video can also be pre-processed, perhaps using a segmentation technique, to reduce the computation and hence latency.

Technical description and context

The Kalman filter is an efficient recursive filter that estimates the internal state of a linear dynamic system from a series of noisy measurements. It is used in a wide range of engineering and econometric applications from radar and computer vision to estimation of structural macroeconomic models,[5][6] and is an important topic in control theory and control systems engineering. Together with the linear-quadratic regulator (LQR), the Kalman filter solves the linear-quadratic-Gaussian control problem (LQG). The Kalman filter, the linear-quadratic regulator and the linear-quadratic-Gaussian controller are solutions to what probably are the most fundamental problems in control theory.

In most applications, the internal state is much larger (more degrees of freedom) than the few "observable" parameters which are measured. However, by combining a series of measurements, the Kalman filter can estimate the entire internal state.

In control theory, the Kalman filter is most commonly referred to as linear quadratic estimation (LQE).

In Dempster-Shafer theory, each state equation or observation is considered a special case of a Linear belief function and the Kalman filter is a special case of combining linear belief functions on a join-tree or Markov tree.

A wide variety of Kalman filters have now been developed, from Kalman's original formulation, now called the "simple" Kalman filter, the Kalman-Bucy filter, Schmidt's "extended" filter, the "information" filter, and a variety of "square-root" filters that were developed by Bierman, Thornton and many others. Perhaps the most commonly used type of very simple Kalman filter is the phase-locked loop, which is now ubiquitous in radios, especially frequency modulation (FM) radios, television sets, satellite communications receivers, outer space communications systems, and nearly any other electronic communications equipment.

Underlying dynamic system model

Kalman filters are based on linear dynamic systems discretized in the time domain. They are modelled on a Markov chain built on linear operators perturbed by Gaussian noise. The state of the system is represented as a vector of real numbers. At each discrete time increment, a linear operator is applied to the state to generate the new state, with some noise mixed in, and optionally some information from the controls on the system if they are known. Then, another linear operator mixed with more noise generates the observed outputs from the true ("hidden") state. The Kalman filter may be regarded as analogous to the hidden Markov model, with the key difference that the hidden state variables take values in a continuous space (as opposed to a discrete state space as in the hidden Markov model). Additionally, the hidden Markov model can represent an arbitrary distribution for the next value of the state variables, in contrast to the Gaussian noise model that is used for the Kalman filter. There is a strong duality between the equations of the Kalman Filter and those of the hidden Markov model. A review of this and other models is given in Roweis and Ghahramani (1999)[7] and Hamilton (1994), Chapter 13.[8]
In order to use the Kalman filter to estimate the internal state of a process given only a sequence of noisy observations, one must model the process in accordance with the framework of the Kalman filter. This means specifying the following matrices: $F_k$, the state-transition model; $H_k$, the observation model; $Q_k$, the covariance of the process noise; $R_k$, the covariance of the observation noise; and sometimes $B_k$, the control-input model, for each time-step, $k$, as described below.

The Kalman filter model assumes the true state at time $k$ is evolved from the state at $(k-1)$ according to

$$\mathbf{x}_k = F_k \mathbf{x}_{k-1} + B_k \mathbf{u}_k + \mathbf{w}_k$$

where
- $F_k$ is the state transition model which is applied to the previous state $\mathbf{x}_{k-1}$;
- $B_k$ is the control-input model which is applied to the control vector $\mathbf{u}_k$;
- $\mathbf{w}_k$ is the process noise which is assumed to be drawn from a zero mean multivariate normal distribution with covariance $Q_k$.

$$\mathbf{w}_k \sim N(0, Q_k)$$

At time $k$ an observation (or measurement) $\mathbf{z}_k$ of the true state $\mathbf{x}_k$ is made according to

$$\mathbf{z}_k = H_k \mathbf{x}_k + \mathbf{v}_k$$

where $H_k$ is the observation model which maps the true state space into the observed space and $\mathbf{v}_k$ is the observation noise which is assumed to be zero mean Gaussian white noise with covariance $R_k$.

$$\mathbf{v}_k \sim N(0, R_k)$$

The initial state, and the noise vectors at each step $\{\mathbf{x}_0, \mathbf{w}_1, ..., \mathbf{w}_k, \mathbf{v}_1, ..., \mathbf{v}_k\}$ are all assumed to be mutually independent.

Many real dynamical systems do not exactly fit this model. In fact, unmodelled dynamics can seriously degrade the filter performance, even when it was supposed to work with unknown stochastic signals as inputs. The reason for this is that the effect of unmodelled dynamics depends on the input, and, therefore, can bring the estimation algorithm to instability (it diverges). On the other hand, independent white noise signals will not make the algorithm diverge. The problem of separating between measurement noise and unmodelled dynamics is a difficult one and is treated in control theory under the framework of robust control.
The Kalman filter

The Kalman filter is a recursive estimator. This means that only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state. In contrast to batch estimation techniques, no history of observations and/or estimates is required. In what follows, the notation $\hat{x}_{n|m}$ represents the estimate of $\mathbf{x}$ at time $n$ given observations up to, and including at time $m$.

The state of the filter is represented by two variables:
- $\hat{x}_{k|k}$, the a posteriori state estimate at time $k$ given observations up to and including at time $k$;
- $P_{k|k}$, the a posteriori error covariance matrix (a measure of the estimated accuracy of the state estimate).

The Kalman filter can be written as a single equation, however it is most often conceptualized as two distinct phases: "Predict" and "Update". The predict phase uses the state estimate from the previous timestep to produce an estimate of the state at the current timestep. This predicted state estimate is also known as the a priori state estimate because, although it is an estimate of the state at the current timestep, it does not include observation information from the current timestep. In the update phase, the current a priori prediction is combined with current observation information to refine the state estimate. This improved estimate is termed the a posteriori state estimate.

Typically, the two phases alternate, with the prediction advancing the state until the next scheduled observation, and the update incorporating the observation. However, this is not necessary; if an observation is unavailable for some reason, the update may be skipped and multiple prediction steps performed. Likewise, if multiple independent observations are available at the same time, multiple update steps may be performed (typically with different observation matrices $H_k$).

Predict

Predicted (a priori) state estimate

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k u_k$$

Predicted (a priori) estimate covariance

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

Update

Innovation or measurement residual

$$\tilde{y}_k = z_k - H_k \hat{x}_{k|k-1}$$

Innovation (or residual) covariance

$$S_k = H_k P_{k|k-1} F_k^T + R_k$$

Optimal Kalman gain

$$K_k = P_{k|k-1} F_k^T S_k^{-1}$$

Updated (a posteriori) state estimate

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$

Updated (a posteriori) estimate covariance

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

The formula for the updated estimate and covariance above is only valid for the optimal Kalman gain. Usage of other gain values require a more complex formula found in the derivations section.
Invariants

If the model is accurate, and the values for $\hat{x}_0|0$ and $P_{0|0}$ accurately reflect the distribution of the initial state values, then the following invariants are preserved: (all estimates have mean error zero)

- $E[x_k - \hat{x}_k|k] = E[x_k - \hat{x}_k|k-1] = 0$
- $E[\hat{y}_k] = 0$

where $E[\xi]$ is the expected value of $\xi$, and covariance matrices accurately reflect the covariance of estimates

- $P_{k|k} = \text{cov}(x_k - \hat{x}_k|k)$
- $P_{k|k-1} = \text{cov}(x_k - \hat{x}_k|k-1)$
- $S_k = \text{cov}(\hat{y}_k)$

Example application, technical

Consider a truck on perfectly frictionless, infinitely long straight rails. Initially the truck is stationary at position 0, but it is buffeted this way and that by random acceleration. We measure the position of the truck every $\Delta t$ seconds, but these measurements are imprecise; we want to maintain a model of where the truck is and what its velocity is.

We show here how we derive the model from which we create our Kalman filter.

Since $F$, $H$, $R$ and $Q$ are constant, their time indices are dropped.

The position and velocity of the truck are described by the linear state space

$$x_k = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

where $\dot{x}$ is the velocity, that is, the derivative of position with respect to time.

We assume that between the $(k-1)$ and $k$ timestep the truck undergoes a constant acceleration of $a_k$ that is normally distributed, with mean 0 and standard deviation $\sigma_a$. From Newton's laws of motion we conclude that

$$x_k = Fx_{k-1} + G a_k$$

(note that there is no $B_u$ term since we have no known control inputs) where

$$F = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix}$$

so that

$$x_k = Fx_{k-1} + w_k$$

where $w_k \sim N(0, Q)$ and

$$Q = GG^T \sigma_a^2 = \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} \\ \frac{\Delta t^3}{2} & \Delta t^2 \end{bmatrix} \sigma_a^2.$$

At each time step, a noisy measurement of the true position of the truck is made. Let us suppose the measurement noise $v_k$ is also normally distributed, with mean 0 and standard deviation $\sigma_z$.

$$z_k = Hx_k + v_k$$

where

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$R = E[v_kv_k^T] = \sigma_z^2$$

Source URL: http://www.en.wikipedia.org/wiki/Kalman_filter
Saylor URL: http://www.saylor.org/courses/cs408
Attributed to [Wikipedia]
We know the initial starting state of the truck with perfect precision, so we initialize

\[
\hat{x}_{0|0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

and to tell the filter that we know the exact position, we give it a zero covariance matrix:

\[
P_{0|0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

If the initial position and velocity are not known perfectly the covariance matrix should be initialized with a suitably large number, say \(L\), on its diagonal.

\[
P_{0|0} = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}
\]

The filter will then prefer the information from the first measurements over the information already in the model.

**Derivations**

**Deriving the *a posteriori* estimate covariance matrix**

Starting with our invariant on the error covariance \(P_{k|k}\) as above

\[
P_{k|k} = \text{cov}(x_k - \hat{x}_{k|k})
\]

substitute in the definition of \(\hat{x}_{k|k}\)

\[
P_{k|k} = \text{cov}(x_k - (\hat{x}_{k|k-1} + K_k \hat{y}_k))
\]

and substitute \(\hat{y}_k\)

\[
P_{k|k} = \text{cov}(x_k - (\hat{x}_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1})))
\]

and \(z_k\)

\[
P_{k|k} = \text{cov}(x_k - (\hat{x}_{k|k-1} + K_k (H_k x_k + v_k - H_k \hat{x}_{k|k-1})))
\]

by collecting the error vectors we get

\[
P_{k|k} = \text{cov}((I - K_k H_k)(x_k - \hat{x}_{k|k-1}) - K_k v_k)
\]

Since the measurement error \(v_k\) is uncorrelated with the other terms, this becomes

\[
P_{k|k} = \text{cov}((I - K_k H_k)(x_k - \hat{x}_{k|k-1})) + \text{cov}(K_k v_k)
\]

by the properties of vector covariance this becomes

\[
P_{k|k} = (I - K_k H_k)\text{cov}(x_k - \hat{x}_{k|k-1})(I - K_k H_k)^T + K_k \text{cov}(v_k)K_k^T
\]

which, using our invariant on \(P_{k|k-1}\) and the definition of \(R_k\) becomes

\[
P_{k|k} = (I - K_k H_k)P_{k|k-1}(I - K_k H_k)^T + K_k R_k K_k^T
\]

This formula (sometimes known as the "Joseph form" of the covariance update equation) is valid for any value of \(K_k\). It turns out that if \(K_k\) is the optimal Kalman gain, this can be simplified further as shown below.

**Kalman gain derivation**

The Kalman filter is a minimum mean-square error estimator. The error in the *a posteriori* state estimation is

\[
x_k - \hat{x}_{k|k}
\]

We seek to minimize the expected value of the square of the magnitude of this vector, \(E[|x_k - \hat{x}_{k|k}|^2]\). This is equivalent to minimizing the trace of the *a posteriori* estimate covariance matrix \(P_{k|k}\). By expanding out the terms in the equation above and collecting, we get:

\[
P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} H_k^T K_k^T + K_k (H_k P_{k|k-1} H_k^T + R_k) K_k^T
\]
\[ P_{k|k-1} = K_k H_k P_{k|k-1} H_k^T K_k^T + K_k S_k K_k^T \]

The trace is minimized when the matrix derivative is zero:

\[ \frac{\partial \text{tr}(P_{k|k})}{\partial K_k} = -2(H_k P_{k|k-1})^T + 2K_k S_k = 0 \]

Solving this for \( K_k \) yields the Kalman gain:

\[ K_k S_k = (H_k P_{k|k-1})^T = P_{k|k-1} H_k^T \]
\[ K_k = \frac{P_{k|k-1} H_k^T}{S_k} \]

This gain, which is known as the \textit{optimal Kalman gain}, is the one that yields MMSE estimates when used.

### Simplification of the a posteriori error covariance formula

The formula used to calculate the \textit{a posteriori} error covariance can be simplified when the Kalman gain equals the optimal value derived above. Multiplying both sides of our Kalman gain formula on the right by \( S_k K_k^T \), it follows that

\[ K_k S_k K_k^T = P_{k|k-1} H_k^T K_k^T \]

Referring back to our expanded formula for the \textit{a posteriori} error covariance,

\[ P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} - P_{k|k-1} H_k^T K_k + K_k S_k K_k^T \]

we find the last two terms cancel out, giving

\[ P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} = (I - K_k H_k) P_{k|k-1}. \]

This formula is computationally cheaper and thus nearly always used in practice, but is only correct for the optimal gain. If arithmetic precision is unusually low causing problems with numerical stability, or if a non-optimal Kalman gain is deliberately used, this simplification cannot be applied; the \textit{a posteriori} error covariance formula as derived above must be used.

### Sensitivity analysis

The Kalman filtering equations provide an estimate of the state \( \hat{x}_{k|k} \) and its error covariance \( P_{k|k} \) recursively. The estimate and its quality depend on the system parameters and the noise statistics fed as inputs to the estimator. This section analyzes the effect of uncertainties in the statistical inputs to the filter.[9] In the absence of reliable statistics or the true values of noise covariance matrices \( Q_k \) and \( R_k \), the expression

\[ P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^T + K_k R_k K_k \]

no longer provides the actual error covariance. In other words, \( P_{k|k} \neq E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T] \). In most real time applications the covariance matrices that are used in designing the Kalman filter are different from the actual noise covariances matrices. This sensitivity analysis describes the behavior of the estimation error covariance when the noise covariances as well as the system matrices \( F_k \) and \( H_k \) that are fed as inputs to the filter are incorrect. Thus, the sensitivity analysis describes the robustness (or sensitivity) of the estimator to misspecified statistical and parametric inputs to the estimator.

\[ Q_k^a \quad R_k^a \]
\[ Q_k^a \equiv P_{k|k}^a = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T] \]
\[ P_{k|k} = P_k^a \]
\[ E[w_k w_k^T] = Q_k^a \]
\[ E[v_k v_k^T] = R_k^a \]
\[ \hat{x}_{k|k} \]

Source URL: http://www.en.wikipedia.org/wiki/Kalman_filter
Saylor URL: http://www.saylor.org/courses/cs408
While computing $P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + R_k$

$P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^T + K_k R_k K_k$

- While computing $P_{k|k}$, by design the filter implicitly assumes that $E[w_k w_k^T] = Q_k$ and $E[v_k v_k^T] = R_k$.
- The recursive expressions for $P_{k|k}$ and $P_{k|k}$ are identical except for the presence of $Q_k$ and $R_k$ in place of the design values $Q_k$ and $R_k$ respectively.

**Square root form**

One problem with the Kalman filter is its numerical stability; when the process is well known (the process noise covariance $Q_k$ is small), it is easy for rounding error to render the state covariance matrix $P$ invalid: negative diagonal entries or otherwise not positive semi-definite.

Positive definite matrices have the property that they have a triangular matrix square root $P = S S^T$. This can be computed efficiently using the Cholesky factorization algorithm, but more importantly if the covariance is kept in this form, it can never have a negative diagonal or become asymmetric. An equivalent form, which avoids many of the square root operations required by the matrix square root yet preserves the desirable numerical properties, is the $U-D$ decomposition form, $P = U D U^T$, where $U$ is a unit triangular matrix (with unit diagonal), and $D$ is a diagonal matrix.

Between the two, the U-D factorization uses the same amount of storage, and somewhat less computation, and is the most commonly used square root form. (Early literature on the relative efficiency is somewhat misleading, as it assumed that square roots were much more time-consuming than divisions,[10]:69 while on 21-st century computers they are only slightly more expensive.)

Efficient algorithms for the Kalman prediction and update steps in the square root form were developed by G. J. Bierman and C. L. Thornton.[10][11]

The $L D L^T$ decomposition of the innovation covariance matrix $S_k$ is the basis for another type of numerically efficient and robust square root filter.[12] The algorithm starts with the LU decomposition as implemented in the Linear Algebra PACKage (LAPACK). These results are further factored into the $L D L^T$ structure with methods given by Golub and Van Loan (algorithm 4.1.2) for a symmetric nonsingular matrix.[13] Any singular covariance matrix is pivoted so that the first diagonal partition is nonsingular and well-conditioned. The pivoting algorithm must retain any portion of the innovation covariance matrix directly corresponding to observed state-variables $H_k x_{k|k-1}$ that are associated with auxiliary observations in $y_k$. The $L D L^T$ square-root filter requires orthogonalization of the observation vector.[11][12] This may be done with the inverse square-root of the covariance matrix for the auxiliary variables using Method 2 in Higham (2002, p. 263).[14]

**Relationship to recursive Bayesian estimation**

The Kalman filter can be considered to be one of the most simple dynamic Bayesian networks. The Kalman filter calculates estimates of the true values of measurements recursively over time using incoming measurements and a mathematical process model. Similarly, recursive Bayesian estimation calculates estimates of an unknown probability density function (PDF) recursively over time using incoming measurements and a mathematical process model.[15]

In recursive Bayesian estimation, the true state is assumed to be an unobserved Markov process, and the measurements are the observed states of a hidden Markov model (HMM).
Because of the Markov assumption, the true state is conditionally independent of all earlier states given the immediately previous state.

\[ p(x_k | x_0, \ldots, x_{k-1}) = p(x_k | x_{k-1}) \]

Similarly the measurement at the \( k \)-th timestep is dependent only upon the current state and is conditionally independent of all other states given the current state.

\[ p(z_k | x_0, \ldots, x_k) = p(z_k | x_k) \]

Using these assumptions the probability distribution over all states of the hidden Markov model can be written simply as:

\[ p(x_0, \ldots, x_k, z_1, \ldots, z_k) = p(x_0) \prod_{i=1}^{k} p(z_i | x_i) p(x_i | x_{i-1}) \]

However, when the Kalman filter is used to estimate the state \( x \), the probability distribution of interest is that associated with the current states conditioned on the measurements up to the current timestep. This is achieved by marginalizing out the previous states and dividing by the probability of the measurement set.

This leads to the **predict** and **update** steps of the Kalman filter written probabilistically. The probability distribution associated with the predicted state is the sum (integral) of the products of the probability distribution associated with the transition from the \( (k - 1) \)-th timestep to the \( k \)-th and the probability distribution associated with the previous state, over all possible \( z_{k-1} \).

\[ p(x_k | Z_{k-1}) = \int p(x_k | x_{k-1}) p(x_{k-1} | Z_{k-1}) \, dx_{k-1} \]

The measurement set up to time \( t \) is

\[ Z_t = \{ z_1, \ldots, z_t \} \]

The probability distribution of the update is proportional to the product of the measurement likelihood and the predicted state.

\[ p(x_k | Z_k) = \frac{p(z_k | x_k) p(x_k | Z_{k-1})}{p(z_k | Z_{k-1})} \]

The denominator

\[ p(z_k | Z_{k-1}) = \int p(z_k | x_k) p(x_k | Z_{k-1}) \, dx_k \]

is a normalization term.
The remaining probability density functions are
\[ p(x_k|x_{k-1}) = \mathcal{N}(F_k x_{k-1}, Q_k) \]
\[ p(z_k|x_k) = \mathcal{N}(H_k x_k, R_k) \]
\[ p(x_{k-1}|z_{k-1}) = \mathcal{N}(\hat{x}_{k-1}, P_{k-1}) \]
Note that the PDF at the previous timestep is inductively assumed to be the estimated state and covariance. This is justified because, as an optimal estimator, the Kalman filter makes best use of the measurements, therefore the PDF for \( x_k \) given the measurements \( Z_k \) is the Kalman filter estimate.

**Information filter**

In the information filter, or inverse covariance filter, the estimated covariance and estimated state are replaced by the information matrix and information vector respectively. These are defined as:

\[
Y_{k|k} = P_{k|k}^{-1} \\
\hat{y}_{k|k} = P_{k|k}^{-1} \hat{x}_{k|k}
\]

Similarly the predicted covariance and state have equivalent information forms, defined as:

\[
Y_{k|k-1} = P_{k|k-1}^{-1} \\
\hat{y}_{k|k-1} = P_{k|k-1}^{-1} \hat{x}_{k|k-1}
\]
as have the measurement covariance and measurement vector, which are defined as:

\[
I_k = H_k^T R_k^{-1} H_k \\
i_k = H_k^T R_k^{-1} z_k
\]
The information update now becomes a trivial sum.

\[
Y_{k|k} = Y_{k|k-1} + I_k \\
\hat{y}_{k|k} = \hat{y}_{k|k-1} + i_k
\]
The main advantage of the information filter is that \( N \) measurements can be filtered at each timestep simply by summing their information matrices and vectors.

\[
Y_{k|k} = Y_{k|k-1} + \sum_{j=1}^{N} I_{k,j} \\
\hat{y}_{k|k} = \hat{y}_{k|k-1} + \sum_{j=1}^{N} i_{k,j}
\]
To predict the information filter the information matrix and vector can be converted back to their state space equivalents, or alternatively the information space prediction can be used.

\[
M_k = [F_k^{-1}]^T Y_{k-1|k-1} F_k^{-1} \\
C_k = M_k [M_k + Q_k^{-1}]^{-1} \\
I_k = I - C_k \\
Y_{k|k-1} = I_k M_k I_k^T + C_k Q_k^{-1} C_k^T \\
\hat{y}_{k|k-1} = I_k [F_k^{-1}]^T \hat{y}_{k-1|k-1}
\]
Note that if \( F \) and \( Q \) are time invariant these values can be cached. Note also that \( F \) and \( Q \) need to be invertible.
**Fixed-lag smoother**

The optimal fixed-lag smoother provides the optimal estimate of \( \hat{x}_{k-N|k} \) for a given fixed-lag \( N \) using the measurements from \( z_1 \) to \( z_k \). It can be derived using the previous theory via an augmented state, and the main equation of the filter is the following:

\[
\begin{bmatrix}
\hat{x}_{t|t} \\
\hat{x}_{t-1|t} \\
\vdots \\
\hat{x}_{t-N+1|t}
\end{bmatrix} =
\begin{bmatrix}
I \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{t|t-1} \\
\hat{x}_{t-1|t-1} \\
\vdots \\
\hat{x}_{t-N|t-1}
\end{bmatrix} +
\begin{bmatrix}
0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{t-1|t-1} \\
\hat{x}_{t-2|t-1} \\
\vdots \\
\hat{x}_{t-N|t-1}
\end{bmatrix} +
\begin{bmatrix} K^{(1)} \\ K^{(2)} \\ \vdots \\ K^{(N)} \end{bmatrix} y_{t|t-1}
\]

where:
- \( \hat{x}_{t|t-1} \) is estimated via a standard Kalman filter;
- \( y_{t|t-1} = z(t) - \hat{x}_{t|t-1} \) is the innovation produced considering the estimate of the standard Kalman filter;
- the various \( \hat{x}_{t-i|t} \) with \( i = 0, \ldots, N \) are new variables, i.e. they do not appear in the standard Kalman filter;
- the gains are computed via the following scheme:

\[
K^{(i)} = P^{(i)} H^{T} \left( H P H^{T} + R \right)^{-1}
\]

and

\[
P^{(i)} = P \left[ (F - K H)^{T} \right]^{i}
\]

where \( P \) and \( K \) are the prediction error covariance and the gains of the standard Kalman filter.

If the estimation error covariance is defined so that

\[
P_{i} := E \left[ (x_{t-i} - \hat{x}_{t-i|t})^{*} (x_{t-i} - \hat{x}_{t-i|t}) | z_{1} \ldots z_{t} \right],
\]

then we have that the improvement on the estimation of \( x_{t-i} \) is given by:

\[
P - P_{i} = \sum_{j=0}^{i} \left[ P^{(j)} H^{T} \left( H P H^{T} + R \right)^{-1} H \left( P^{(i)} \right)^{T} \right]
\]

**Fixed-interval smoothers**

The optimal fixed-interval smoother provides the optimal estimate of \( \hat{x}_{k|n} \) (\( k < n \)) using the measurements from a fixed interval \( z_1 \) to \( z_n \). This is also called "Kalman Smoothing". There are several smoothing algorithms in common use.

**Rauch–Tung–Striebel**

The Rauch–Tung–Striebel (RTS) Smoother\(^{[16]}\) is an efficient two-pass algorithm for fixed interval smoothing. The main equations of the smoother are the following (assuming \( B_{k} = 0 \)):

- forward pass: regular Kalman filter algorithm
- backward pass:

\[
\hat{x}_{k|n} = \mathbf{F}_{k}^{-1} \mathbf{Q}_{k} \mathbf{P}_{k+1|k}^{-1}, \quad \text{where}
\]

\[
\mathbf{F}_{k}^{-1}(1 - \mathbf{Q}_{k} \mathbf{P}_{k+1|k}^{-1})
\]

\[
\mathbf{K}_{k} = \mathbf{F}_{k}^{-1} \mathbf{Q}_{k} \mathbf{P}_{k+1|k}^{-1}
\]
Modified Bryson-Frazier Smoother

An alternative to the RTS algorithm is the Modified Bryson-Frazier (MBF) fixed interval smoother developed by Bierman.[11] This also uses a backward pass that processes data saved from the Kalman filter forward pass. The equations for the backward pass involve the recursive computation of data which are used at each observation time to compute the smoothed state and covariance.

The recursive equations are

\[
\hat{\lambda}_k = H_k^T S_k^{-1} H_k + C_k^T \hat{\lambda}_k C_k
\]

\[
\hat{\lambda}_{k-1} = F_k^T \hat{\lambda}_k F_k
\]

\[
\hat{\lambda}_n = 0
\]

\[
\hat{\lambda}_k = -H_k^T S_k^{-1} z_k + C_k^T \hat{\lambda}_k
\]

\[
\hat{\lambda}_{k-1} = F_k^T \hat{\lambda}_k
\]

\[
\hat{\lambda}_n = 0
\]

where \( S_k \) is the residual covariance and \( C_k = I - K_k H_k \). The smoothed state and covariance can then be found by substitution in the equations

\[
P_{k|n} = P_{k|k} - P_{k|k} \hat{\lambda}_k P_{k|k}
\]

\[
x_{k|n} = x_{k|k} - P_{k|k} \hat{\lambda}_k
\]

or

\[
P_{k|n} = P_{k|k-1} - P_{k|k-1} \hat{\lambda}_k P_{k|k-1}
\]

\[
x_{k|n} = x_{k|k-1} - P_{k|k-1} \hat{\lambda}_k
\]

An important advantage of the MBF is that it does not require finding the inverse of the covariance matrix.

Non-linear filters

The basic Kalman filter is limited to a linear assumption. More complex systems, however, can be nonlinear. The non-linearity can be associated either with the process model or with the observation model or with both.

Extended Kalman filter

In the extended Kalman filter (EKF), the state transition and observation models need not be linear functions of the state but may instead be (differentiable) functions.

\[
x_k = f(x_{k-1}, u_k) + w_k
\]

\[
z_k = h(x_k) + v_k
\]

The function \( f \) can be used to compute the predicted state from the previous estimate and similarly the function \( h \) can be used to compute the predicted measurement from the predicted state. However, \( f \) and \( h \) cannot be applied to the covariance directly. Instead a matrix of partial derivatives (the Jacobian) is computed.

At each timestep the Jacobian is evaluated with current predicted states. These matrices can be used in the Kalman filter equations. This process essentially linearizes the non-linear function around the current estimate.
Unscented Kalman filter

When the state transition and observation models – that is, the predict and update functions \( f \) and \( h \) (see above) – are highly non-linear, the extended Kalman filter can give particularly poor performance.\(^{[17]}\) This is because the covariance is propagated through linearization of the underlying non-linear model. The unscented Kalman filter (UKF)\(^{[17]}\) uses a deterministic sampling technique known as the unscented transform to pick a minimal set of sample points (called sigma points) around the mean. These sigma points are then propagated through the non-linear functions, from which the mean and covariance of the estimate are then recovered. The result is a filter which more accurately captures the true mean and covariance. (This can be verified using Monte Carlo sampling or through a Taylor series expansion of the posterior statistics.) In addition, this technique removes the requirement to explicitly calculate Jacobians, which for complex functions can be a difficult task in itself (i.e., requiring complicated derivatives if done analytically or being computationally costly if done numerically).

Predict

As with the EKF, the UKF prediction can be used independently from the UKF update, in combination with a linear (or indeed EKF) update, or vice versa.

The estimated state and covariance are augmented with the mean and covariance of the process noise.

\[
\begin{align*}
X^a_{k-1|k-1} &= [x^T_{k-1|k-1} \quad E[w^T_k]]^T \\
P^a_{k-1|k-1} &= \begin{bmatrix} P_{k-1|k-1} & 0 \\ 0 & Q_k \end{bmatrix}
\end{align*}
\]

A set of \( 2L+1 \) sigma points is derived from the augmented state and covariance where \( L \) is the dimension of the augmented state.

\[
\begin{align*}
X^0_{k-1|k-1} &= x^a_{k-1|k-1} \\
X^i_{k-1|k-1} &= x^a_{k-1|k-1} + (\sqrt{(L+\lambda)P^a_{k-1|k-1}})_i & i = 1..L \\
X^i_{k-1|k-1} &= x^a_{k-1|k-1} - (\sqrt{(L+\lambda)P^a_{k-1|k-1}})_i & i = L+1,\ldots 2L
\end{align*}
\]

where

\[
(\sqrt{(L+\lambda)P^a_{k-1|k-1}})_i
\]

is the \( i \)th column of the matrix square root of

\[
(L + \lambda)P^a_{k-1|k-1}
\]

using the definition: square root \( A \) of matrix \( B \) satisfies

\[
B \equiv AA^T.
\]

The matrix square root should be calculated using numerically efficient and stable methods such as the Cholesky decomposition.

The sigma points are propagated through the transition function \( f \).

\[
\begin{align*}
\hat{X}^i_{k|k-1} &= f(X^i_{k-1|k-1}) & i = 0..2L
\end{align*}
\]

where \( f : \mathbb{R}^L \rightarrow \mathbb{R}^{|X|} \). The weighted sigma points are recombined to produce the predicted state and covariance.

\[
\hat{x}_{k|k-1} = \sum_{i=0}^{2L} W_i^i \hat{X}^i_{k|k-1}
\]
where the weights for the state and covariance are given by:

\[
W_s^0 = \frac{\lambda}{L + \lambda}, \\
W_c^0 = \frac{\lambda}{L + \lambda} + (1 - \alpha^2 + \beta), \\
W_s^i = W_c^i = \frac{1}{2(L + \lambda)}, \\
\lambda = \alpha^2(L + \kappa) - \lambda.
\]

\(\alpha\) and \(\kappa\) control the spread of the sigma points. \(\beta\) is related to the distribution of \(\mathbf{x}\). Normal values are \(\alpha = 10^{-3}\), \(\kappa = 1\) and \(\beta = 2\). If the true distribution of \(\mathbf{x}\) is Gaussian, \(\beta = 2\) is optimal.\(^{[18]}\)

**Update**

The predicted state and covariance are augmented as before, except now with the mean and covariance of the measurement noise.

\[
\mathbf{x}_{k|k-1}^a = [\hat{\mathbf{x}}_{k|k-1}^T \quad E[\mathbf{v}_k^T]]^T
\]

\[
\mathbf{P}_{k|k-1}^a = \begin{bmatrix} \mathbf{P}_{k|k-1} & 0 \\ 0 & \mathbf{R}_k \end{bmatrix}
\]

As before, a set of \(2L + 1\) sigma points is derived from the augmented state and covariance where \(L\) is the dimension of the augmented state.

\[
\begin{align*}
\chi_{k|k-1}^0 &= \mathbf{x}_{k|k-1}^a \\
\chi_{k|k-1}^i &= \mathbf{x}_{k|k-1}^a + \sqrt{(L+\lambda)\mathbf{P}_{k|k-1}^a}, \quad i = 1..L \\
\chi_{k|k-1}^i &= \mathbf{x}_{k|k-1}^a - \sqrt{(L+\lambda)\mathbf{P}_{k|k-1}^a}, \quad i = L+1,..,2L
\end{align*}
\]

Alternatively if the UKF prediction has been used the sigma points themselves can be augmented along the following lines

\[
\chi_{k|k-1} := [\chi_{k|k-1}^T \quad E[\mathbf{v}_k^T]]^T \pm \sqrt{(L+\lambda)\mathbf{R}_k^a}
\]

where

\[
\mathbf{R}_k^a = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{R}_k \end{bmatrix}
\]

The sigma points are projected through the observation function \(h\).

\[
\gamma_k^i = h(\chi_{k|k-1}^i) \quad i = 0..2L
\]

The weighted sigma points are recombined to produce the predicted measurement and predicted measurement covariance.

\[
\hat{\mathbf{z}}_k = \sum_{i=0}^{2L} W_s^i \gamma_k^i
\]

\[
\mathbf{P}_{z_kz_k} = \sum_{i=0}^{2L} W_c^i [\gamma_k^i - \hat{z}_k] [\gamma_k^i - \hat{z}_k]^T
\]

The state-measurement cross-covariance matrix,

\[
\mathbf{P}_{z_kx_k} = \sum_{i=0}^{2L} W_c^i [\chi_{k|k-1}^i - \hat{\mathbf{x}}_{k|k-1}] [\gamma_k^i - \hat{z}_k]^T
\]

Source URL: http://www.en.wikipedia.org/wiki/Kalman_filter
Saylor URL: http://www.saylor.org/courses/cs408
is used to compute the UKF Kalman gain.

\[ K_k = P_{x_k|x_k}^{-1} \]

As with the Kalman filter, the updated state is the predicted state plus the innovation weighted by the Kalman gain,

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - \hat{z}_k) \]

And the updated covariance is the predicted covariance, minus the predicted measurement covariance, weighted by the Kalman gain.

\[ P_{k|k} = P_{k|k-1} - K_k P_{x_k|x_k} K_k^T \]

**Kalman–Bucy filter**

The Kalman–Bucy filter is a continuous time version of the Kalman filter.\(^{[19]}\)\(^{[20]}\)

It is based on the state space model

\[
\begin{align*}
\frac{d}{dt} x(t) &= F(t)x(t) + w(t) \\
\frac{d}{dt} z(t) &= H(t)x(t) + v(t)
\end{align*}
\]

where the covariances of the noise terms \( w(t) \) and \( v(t) \) are given by \( Q(t) \) and \( R(t) \), respectively.

The filter consists of two differential equations, one for the state estimate and one for the covariance:

\[
\begin{align*}
\frac{d}{dt} \hat{x}(t) &= F(t)\hat{x}(t) + K(t)(z(t) - H(t)\hat{x}(t)) \\
\frac{d}{dt} P(t) &= F(t)P(t) + P(t)F^T(t) + Q(t) - K(t)R(t)K^T(t)
\end{align*}
\]

where the Kalman gain is given by

\[ K(t) = P(t)H^T(t)R^{-1}(t) \]

Note that in this expression for \( K(t) \) the covariance of the observation noise \( R(t) \) represents at the same time the covariance of the prediction error (or innovation) \( \tilde{y}(t) = z(t) - H(t)\hat{x}(t) \); these covariances are equal only in the case of continuous time.\(^{[21]}\)

The distinction between the prediction and update steps of discrete-time Kalman filtering does not exist in continuous time.

The second differential equation, for the covariance, is an example of a Riccati equation.

**Hybrid Kalman filter**

Most physical systems are represented as continuous-time models while discrete-time measurements are frequently taken for state estimation via a digital processor. Therefore, the system model and measurement model are given by

\[
\begin{align*}
\dot{x}(t) &= F(t)x(t) + B(t)u(t) + w(t), \quad w(t) \sim N(0, Q(t)) \\
z_k &= H_k x_k + v_k, \quad v_k \sim N(0, R_k)
\end{align*}
\]

where \( x_k = x(t_k) \).

Initialize

\[ \hat{x}_{0|0} = E[x(t_0)], \quad P_{0|0} = Var[x(t_0)] \]

Predict
The prediction equations are derived from those of continuous-time Kalman filter without update from measurements, i.e., $K(t) = 0$. The predicted state and covariance are calculated respectively by solving a set of differential equations with the initial value equal to the estimate at the previous step.

Update

$$
K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}
$$

$$
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1})
$$

$$
P_{k|k} = (I - K_k H_k) P_{k|k-1}
$$

The update equations are identical to those of discrete-time Kalman filter.

**Applications**

- Attitude and Heading Reference Systems
- Autopilot
- Battery state of charge (SoC) estimation [22][23]
- Brain-computer interface
- Chaotic signals
- Dynamic positioning
- Economics, in particular macroeconomics, time series, and econometrics
- Inertial guidance system
- Radar tracker
- Satellite navigation systems
- Simultaneous localization and mapping
- Speech enhancement
- Weather forecasting
- Navigation Systems
- 3D-Modelling

**References**


Further reading


### External links

- Kalman–Bucy Filter (http://www.eng.tau.ac.il/~liptser/lectures1/lect6.pdf), a good derivation of the Kalman–Bucy Filter
- MIT Video Lecture on the Kalman filter (http://academicearth.org/lectures/dynamic-estimation-kalman-filter-and-square-root-filter)
- An Introduction to the Kalman Filter (http://www.cs.unc.edu/~welch/media/pdf/SIGGRAPH2001_CoursePack_08.pdf), SIGGRAPH 2001 Course, Greg Welch and Gary Bishop
- Kalman filtering chapter (http://www.cs.unc.edu/~welch/media/pdf/maybeck_ch1.pdf) from *Stochastic Models, Estimation, and Control*, vol. 1, by Peter S. Maybeck
- Kalman Filter (http://www.cs.unc.edu/~welch/kalman/) webpage, with lots of links
- Kalman Filtering (http://www.innovatia.com/software/papers/kalman.htm)
- Kalman Filters, thorough introduction to several types, together with applications to Robot Localization (http://www.negenborn.net/kal_loc/)
- Source code for the propeller microprocessor (http://obex.parallax.com/objects/239/): Well documented source code written for the Parallax propeller processor.

Source URL: http://www.en.wikipedia.org/wiki/Kalman_filter
Saylor URL: http://www.saylor.org/courses/cs408
• Matlab Toolbox of Kalman Filtering applied to Simultaneous Localization and Mapping (http://eia.udg.es/~qsalvi/Slam.zip): Vehicle moving in 1D, 2D and 3D


• The Kalman Filter Explained (http://www.tristanfletcher.co.uk/LDS.pdf) A very simple tutorial.

• The Kalman Filter in Reproducing Kernel Hilbert Spaces (http://www.cnel.ufl.edu/~weifeng/publication.htm) A comprehensive introduction.


• Extended Kalman Filters (http://apmonitor.com/wiki/index.php/Main/Background) explained in the context of Simulation, Estimation, Control, and Optimization