WHAT IS A DIFFERENTIAL EQUATION?

An equation of the form
\[ \frac{dy}{dx} = f(x) \]

that has a derivative in it is called a differential equation. Differential equations are an important topic in calculus, engineering, and the sciences. A lot of the equations that you work with in science and engineering are derived from a specific type of differential equation called an initial value problem.

INITIAL VALUE PROBLEM

The problem of finding a function y of x when we know its derivative and its value \( y_0 \) at a particular point \( x_0 \) is called an initial value problem. This problem can be solved in two steps.

1. \( \int dy = \int f(x) \, dx \rightarrow y = F(x) + c \leftarrow \) general solution

2. Using the initial data, plug it into the general solution and solve for c.

EXAMPLE 1: Solve the initial value problem.
\[ \frac{dy}{dx} = 10 - x, \quad y(0) = -1 \]

SOLUTION:

STEP 1:
\[ \frac{dy}{dx} = 10 - x \rightarrow dy = (10 - x) \, dx \]
\[ \int dy = \int (10 - x) \, dx \rightarrow y = 10x - \frac{x^2}{2} + c \]

STEP 2: When \( x = 0 \), \( y = -1 \).
\[ -1 = 10(0) - \frac{0}{2} + c \rightarrow c = -1 \]

SOLUTION: \( y = 10x - \frac{x^2}{2} - 1 \)
EXAMPLE 2: Solve the initial value problem.

\[ \frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0 \]

SOLUTION:

STEP 1:

\[ \frac{dy}{dx} = 9x^2 - 4x + 5 \quad \rightarrow \quad \int dy = \int (9x^2 - 4x + 5) \, dx \]

\[ \int dy = \int (9x^2 - 4x + 5) \, dx \quad \rightarrow \quad y = \frac{9x^3}{3} - \frac{4x^2}{2} + 5x + c \]

STEP 2: When \( x = -1 \), \( y = 0 \).

\[ 0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + c \quad \rightarrow \quad 0 = -3 - 2 - 5 + c \quad \rightarrow \quad c = 10 \]

SOLUTION: \( y = 3x^3 - 2x^2 + 5x + 10 \)

EXAMPLE 3: Solve the initial value problem.

\[ \frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1 \]

SOLUTION:

STEP 1:

\[ \frac{ds}{dt} = \cos t + \sin t \quad \rightarrow \quad ds = (\cos t + \sin t) \, dt \]

\[ \int ds = \int (\cos t + \sin t) \, dt \quad \rightarrow \quad s = \sin t - \cos t + c \]

STEP 2: When \( t = \pi \), \( s = 1 \).

\[ 1 = \sin \pi - \cos \pi + c \quad \rightarrow \quad 1 = 0 - (-1) + c \quad \rightarrow \quad c = 0 \]

SOLUTION: \( s = \sin t - \cos t \)

EXAMPLE 4: Solve the initial value problem.

\[ \frac{d^2 y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1 \]

SOLUTION: We will have to do the two steps twice to find the solution to this initial
value problem. The first time through will give us $y'$ and the second time through will give us $y$.

**STEP 1:**

$$y' = \int (2 - 6x) \, dx \quad \Rightarrow \quad y' = 2x - \frac{6x^2}{2} + c$$

**STEP 2:** When $x = 0$, $y' = 4$.

$$4 = 2(0) - 3(0)^2 + c \quad \Rightarrow \quad c = 4$$

$$y' = 2x - 3x^2 + 4$$

**EXAMPLE 5:** Solve the initial value problem.

Since we are working with the fourth derivative, we will have to go through the two steps four times.

**STEP 1:**

$$y = \int (2x - 3x^2 + 4) \, dx \quad \Rightarrow \quad y = \frac{2x^2}{2} - \frac{3x^3}{3} + 4x + c$$

**STEP 2:** When $x = 0$, $y = 1$.

$$1 = 0^2 - 0^3 + 4(0) + c \quad \Rightarrow \quad c = 1$$

**SOLUTION:** $y = x^2 - x^3 + 4x + 1$

**EXAMPLE 5:** Solve the initial value problem.

$$y^{(4)} = -\sin t + \cos t, \quad y^{(3)}(0) = 7, \quad y''(0) = y'(0) = -1,$$

$$y(0) = 0$$

**SOLUTION:** Since we are working with the fourth derivative, we will have to go through the two steps four times.

**STEP 1:**

$$y''' = \int (-\sin t + \cos t) \, dt \quad \Rightarrow \quad y''' = \cos t + \sin t + c$$

**STEP 2:** When $t = 0$, $y''' = 7$.

$$7 = \cos 0 + \sin 0 + c \quad \Rightarrow \quad 7 = 1 + c \quad \Rightarrow \quad c = 6$$

$$y''' = \cos t + \sin t + 6$$

**STEP 1:**
\[ y'' = \int (\cos t + \sin t + 6t) \, dt \quad \rightarrow \quad y'' = \sin t - \cos t + 6t + c \]

**STEP 2:** When \( t = 0 \), \( y'' = -1 \)

\[-1 = \sin 0 - \cos 0 + 6(0) + c \quad \rightarrow \quad -1 = -1 + c \quad \rightarrow \quad c = 0 \]

\[ y'' = \sin t - \cos t + 6t \]

**STEP 1:**

\[ y' = \int (\sin t - \cos t + 6t) \, dt \quad \rightarrow \quad y' = -\cos t - \sin t + \frac{6t^2}{2} + c \]

**STEP 2:** When \( t = 0 \), \( y' = -1 \).

\[-1 = -\cos 0 - \sin 0 + 3(0)^2 + c \quad \rightarrow -1 = -1 + c \quad \rightarrow c = 0 \]

\[ y' = -\cos t - \sin t + 3t^2 \]

**STEP 1:**

\[ y = \int (-\cos t - \sin t + 3t^2) \, dt \quad \rightarrow \quad y = -\sin t + \cos t + \frac{3t^3}{3} + c \]

**STEP 2:** When \( t = 0 \), \( y = 0 \).

\[ 0 = -\sin 0 + \cos 0 + 0^3 + c \quad \rightarrow \quad 0 = 1 + c \quad \rightarrow \quad c = -1 \]

**SOLUTION:** \( y = -\sin t + \cos t + t^3 - 1 \)

**EXAMPLE 6:** Given the velocity,

\[ v = \frac{ds}{dt} = 32t - 2, \]

and the initial position of the body as \( s(1/2) = 4 \). Find the body's position at time \( t \).

**SOLUTION:**

**STEP 1:**

\[ s = \int (32t - 2) \, dt \quad \rightarrow \quad s = \frac{32t^2}{2} - 2t + c \]
STEP 2: When \( t = 1/2 \), \( s = 4 \).

\[
4 = 16 \left( \frac{1}{2} \right)^2 - 2 \left( \frac{1}{2} \right) + c \quad \rightarrow \quad 4 = 4 - 1 + c \quad \rightarrow \quad c = 1
\]

**SOLUTION:** \( s = 16t^2 - 2t + 1 \)

**EXAMPLE 7:** Given the acceleration, \( a = \frac{d^2 s}{dt^2} = -4 \sin 2t \), initial velocity \( v(0) = 2 \), and the initial position of the body as \( s(0) = -3 \). Find the body's position at time \( t \).

**SOLUTION:**

**STEP 1:** 

\[
v = \int -4 \sin 2t \, dt \quad \rightarrow \quad v = \frac{4 \cos 2t}{2} + c
\]

**STEP 2:** When \( t = 0 \), \( v = 2 \).

\[
2 = 2 \cos 0 + c \quad \rightarrow \quad 2 = 2 + c \quad \rightarrow \quad c = 0
\]

**STEP 1:**

\[
s = \int 2 \cos 2t \, dt \quad \rightarrow \quad s = \frac{2 \sin 2t}{2} + c
\]

**STEP 2:** When \( t = 0 \), \( s = -3 \).

\[
-3 = \sin 0 + c \quad \rightarrow \quad c = -3
\]

**SOLUTION:** \( s = \sin 2t - 3 \)

Work through the examples that I have worked in this set of notes. Notice that the higher order derivatives require you to perform the two steps more often. For an example, if we are starting with the 3 derivative, we will have to do the two steps three times.